Quantum Private Information Retrieval

Private Information Retrieval with Quantum Resources

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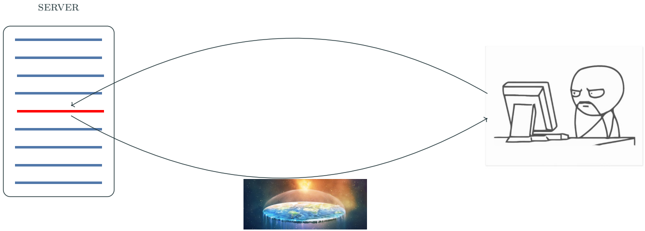
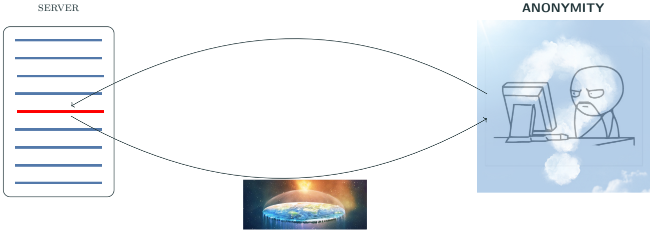
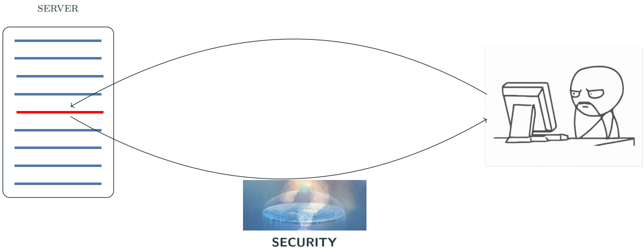
Typical Online Activity



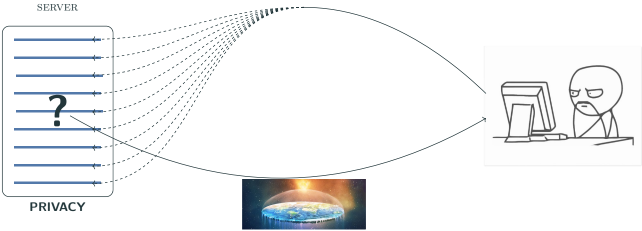
Typical Online Activity: Anonymity



Typical Online Activity: Security



Typical Online Activity: Privacy



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- Increased demand/awareness for privacy.
  - Anonymization, differential privacy, data protection laws ...
- Quest for practical solutions continues.

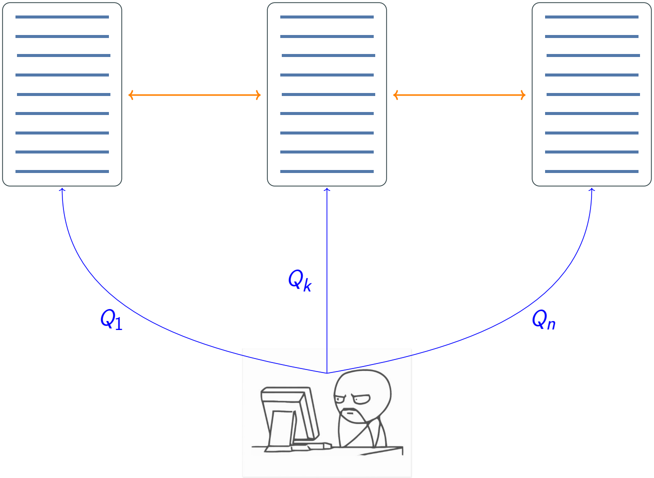
Coded Storage

m files $x^1, \ldots, x^m \in \mathbb{F}_q^{\beta \times k}$ are encoded and stored on n servers by a $[n, k]$ storage code $\mathcal{C}$.

$$\begin{matrix} \text{file } 1 \\ \\ \\ \text{file } m \end{matrix} \begin{pmatrix} x^1_{1,1} & \cdots & x^1_{1,k} \\ \vdots & \ddots & \vdots \\ x^1_{\beta,1} & \cdots & x^1_{\beta,k} \\ \vdots & \vdots & \vdots \\ x^m_{1,1} & \cdots & x^m_{1,k} \\ \vdots & \ddots & \vdots \\ x^m_{\beta,1} & \cdots & x^m_{\beta,k} \end{pmatrix} \cdot \mathbf{G}_{\mathcal{C}} = \begin{pmatrix} y^1_{1,1} & \cdots & y^1_{1,n} \\ \vdots & \ddots & \vdots \\ y^1_{\beta,1} & \cdots & y^1_{\beta,n} \\ \vdots & \vdots & \vdots \\ y^m_{1,1} & \cdots & y^m_{1,n} \\ \vdots & \ddots & \vdots \\ y^m_{\beta,1} & \cdots & y^m_{\beta,n} \end{pmatrix}$$

$\text{SERVER}_1 \quad \text{SERVER}_n$

Private Information Retrieval (PIR)



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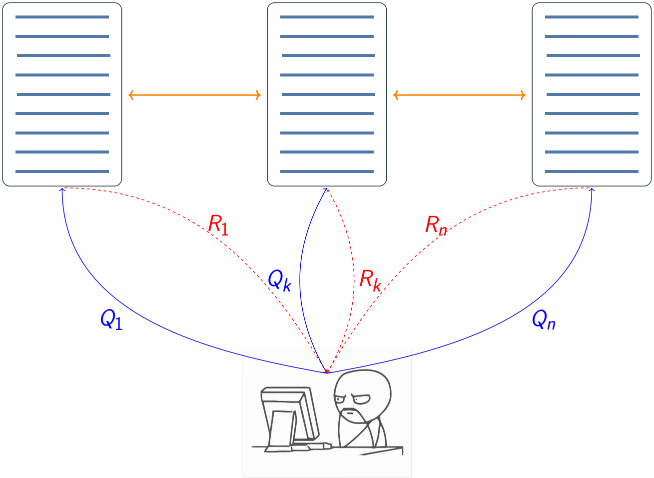

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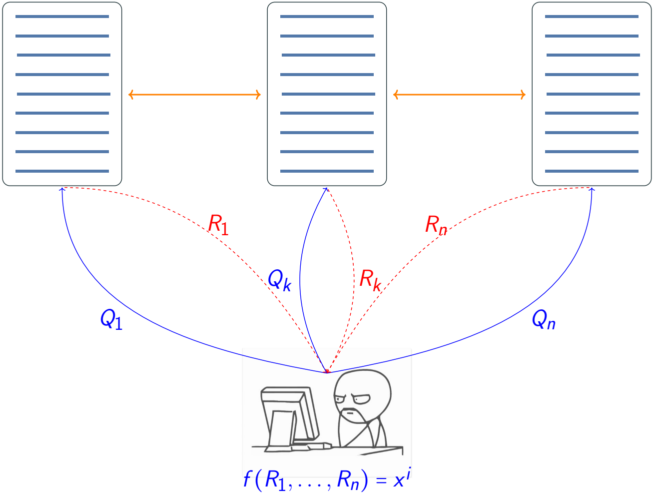

PIR with t-collusion (t-PIR)

Definition (t-PIR).

User privacy: Any set of at most t colluding nodes learns no information about the index i of the desired file, *i.e.*, the mutual information

$$I(i; Q_{\mathcal{T}}^K, R_{\mathcal{T}}^K, y_{\mathcal{T}}) = 0, \quad \forall\ \mathcal{T} \subset [n], |\mathcal{T}| \leq t\ .$$

Server privacy: The user does not learn any information about the files other than the requested one, *i.e.*,

$$I(x^j; Q^K, R^K, K) = 0, \quad \forall j \neq K\ .$$

A scheme with both user and server privacy is called *symmetric*.

PIR with t-collusion (t-PIR)

Definition (Rate and Capacity).

For a PIR scheme the **rate** is the number of information bits of the requested file retrieved per downloaded bits, *i.e.*,

$$R_{\mathrm{PIR}} = \frac{\text{Number of bits in a file}}{\text{Number of downloaded bits}} .$$

The PIR **capacity** is the supremum of PIR rates of all possible PIR schemes, for a fixed parameter setting.

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QPIR is PIR with "*entangled servers*" and "*quantum answers*".

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- Replicated storage with $t = n - 1$ collusion.
- **Goal**: $[n, k]$ coded storage with $t = n - k$ collusion.

Ingredients for QPIR

- Quantum adaptation of existing schemes.
- Generalized Reed-Solomon codes

$$\mathrm{GRS}_k(\alpha, v) = \{(v_i f(\alpha_i))_{1 \le i \le n} \mid f(x) \in \mathbb{F}_q^{<k}[x]\}.$$

- Quantum Computation.

Ingredients for QPIR: Quantum Computation

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- Weyl Operator $\mathbf{W}(a,b) = \mathbf{X}^a \mathbf{Z}^b$.
- **The PVM**
  $\mathcal{B}_{\mathbb{F}_2^2} = \{\mathbf{B}_{(a,b)} = \mathbf{W}_1(a,b)|\Phi\rangle\langle\Phi|\mathbf{W}_1(a,b)^{\mathrm{t}} \mid a,b \in \mathbb{F}_2\}$.

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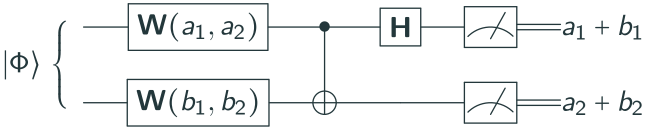
- **Two-Sum Protocol**: Alice and Bob send the sum $(a_1 + b_1, a_2 + b_2)$ of their bits to Carol.



A QPIR Example

- $n = 4$ servers and $[4,2]_4$ - coded database with RS code

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  - $\beta = 1$ and $k = 2$ (determined by encoding).
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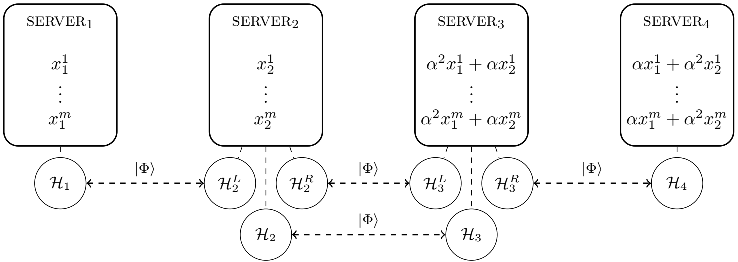
- Query index K, i.e., the requested file is $x^K = (x_1^K, x_2^K)$.

A QPIR Example: Entangled Servers



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$$\begin{aligned}(Q_1, Q_2, Q_3, Q_4) &= (Z_1, Z_2) \cdot \mathbf{G}_{\mathcal{C}^\perp} + \xi_{K,1} \\ &= [Z_1, Z_2, \alpha^2 Z_1 + \alpha Z_2, \alpha Z_1 + \alpha^2 Z_2^2] + \xi_{K,1}.\end{aligned}$$

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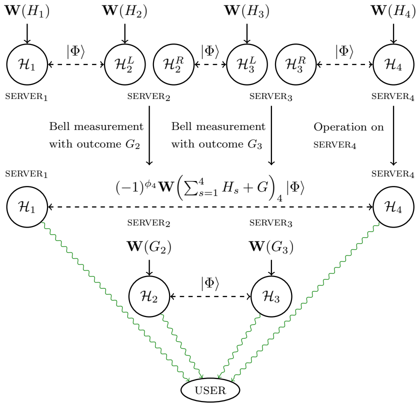
- Query Q_s to server s.

A QPIR Example: Servers' Response



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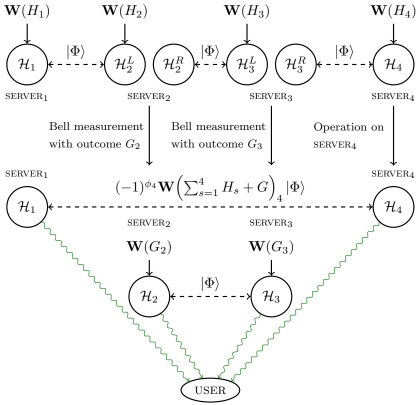

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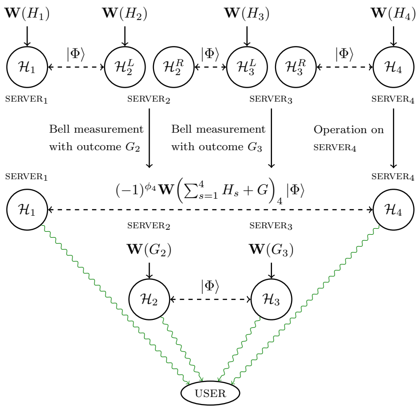

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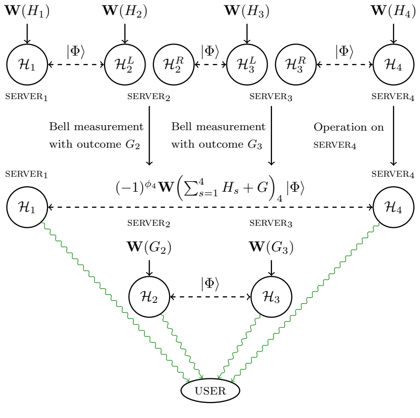

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- Each server sends its qubit to the user.

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Remark

Here we targeted servers 1&2 (systematic encoding). Since the storage is MDS-coded, one can target any two (k in general) servers.

A QPIR Example: Secrecy, Collusion, Rate

- **User secrecy**: queries $Q_1, \ldots, Q_4$ independent of the index K, two random vectors generated and encoded into queries $\Rightarrow$ at least three servers needed in order to retrieve the file requested $\Rightarrow$ **2-collusion**.

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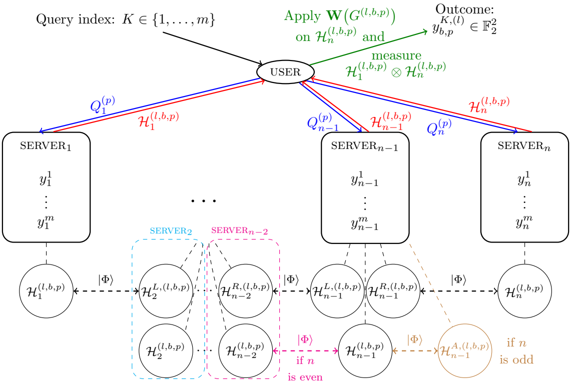
- Server secrecy: obtained for any p because the received state of the user is independent of the fragments x_p^i with $i \neq K$ and the measurement outcome $G^{(p)}$ is independent of any file.

- **Rate**: $R = \frac{2 \cdot 2}{2 \cdot 4} = \frac{1}{2}$.

QPIR with n Servers



$$\text{With } n = k + t, \qquad R_{\mathrm{PIR}} = \begin{cases} \frac{2}{n}, & \text{if } n \text{ is even,} \\ \frac{2}{n+1}, & \text{if } n \text{ is odd,} \end{cases}$$

Ingredients for t-QPIR

Star Product PIR scheme from Freij-Hollanti et al.

- Coded **storage** with storage code $\mathcal{C}$.
- A **retrieval** code $\mathcal{D}$ that determines the privacy.

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Adapted Star Product scheme for t-QPIR

- Weakly self-dual codes.

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Lemma

Let q be even with $q \geq n$. For any $[n, k]$ GRS code $\mathcal{C}$ there exists an $[n, t]$ GRS code $\mathcal{D}$ such that their star-product $\mathcal{S} = \mathcal{C} \star \mathcal{D}$ is an $[n, k + t - 1]$ weakly self-dual GRS code.

Capacity [1]

Capacities	PIR	ref.	SPIR	ref.	QPIR	ref.
Replicated storage, no collusion	$1-\frac{1}{n}$	[3]	$1-\frac{1}{n}$	[6]	1	[21]
Replicated storage, t-collusion	$1-\frac{t}{n}$	[4]	$1-\frac{t}{n}$	[25]	$\min\{1, \frac{2(n-t)}{n}\}$	[23]
$[n,k]$-MDS coded storage, no collusion	$1-\frac{k}{n}$	[5]	$1-\frac{k}{n}$	[7]	$\min\{1, \frac{2(n-k)}{n}\}$	–
$[n,k]$-MDS coded storage, t-collusion	$1-\frac{k+t-1}{n}$	[12]	$1-\frac{k+t-1}{n}$	[7]	$\min\{1, \frac{2(n-k-t+1)}{n}\}$	–

1. M. Allaix, S. Song, L. Holzbaur, T. Pllaha, M. Hayashi, and C. Hollanti. "On the capacity of quantum private information retrieval from MDS-coded and colluding servers," *IEEE Journal on Selected Areas in Communications*, vol. 40, no. 3, pp. 885-898, March 2022.

What's next?

QSDMM: **Q**uantum **S**ecure **D**istributed **M**atrix **M**ultiplication

References:

- M. Allaix, S. Song, L. Holzbaur, **T. Pllaha**, M. Hayashi, and C. Hollanti. "On the capacity of quantum private information retrieval from MDS-coded and colluding servers," *IEEE Journal on Selected Areas in Communications*, vol. 40, no. 3, pp. 885-898, March 2022.
- M. Allaix, L. Holzbaur, **T. Pllaha**, C. Hollanti. "High-Rate Quantum Private Information Retrieval with Weakly Self-Dual Star Product Codes," *In 2021 IEEE International Symposium on Information Theory*, 1046-1051.
- M. Allaix, L. Holzbaur, **T. Pllaha**, C. Hollanti. "Quantum Private Information Retrieval from Coded and Colluding Servers," *IEEE Journal on Selected Areas in Information Theory*, 1(2), 599-610, August 2020.
- M. Allaix, L. Holzbaur, **T. Pllaha**, C. Hollanti. "Quantum Private Information Retrieval from MDS-coded and Colluding Servers," *In 2020 IEEE International Symposium on Information Theory*, 1059–1064.

Thank You!