Laplacian Simplices II: A Coding Theoretic Approach

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AMS Sectional Meetings Ann Arbor, MI

*Joint with Marie Meyer

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1 (Ehrhart) Theory of simplices

2 Laplacian simplices

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1 (Ehrhart) Theory of simplices

2 Laplacian simplices

3 Reflexive Laplacian simplices

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- **1** (Ehrhart) Theory of simplices
- 2 Laplacian simplices
- **3** Reflexive Laplacian simplices
- **4** Further research

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(Ehrhart) Theory of simplices

A simplex Δ in \mathbb{R}^d is a full-dimensional convex hull of $d+1$ points $\mathsf{v}_1,\ldots,\mathsf{v}_{d+1}$ (in \mathbb{R}^d).

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A simplex Δ in \mathbb{R}^d is a full-dimensional convex hull of $d+1$ points $\mathsf{v}_1,\ldots,\mathsf{v}_{d+1}$ (in \mathbb{R}^d). Throughout we will focus on lattice simplices.

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(Ehrhart) Theory of simplices

- A simplex Δ in \mathbb{R}^d is a full-dimensional convex hull of $d+1$ points $\mathsf{v}_1,\ldots,\mathsf{v}_{d+1}$ (in \mathbb{R}^d). Throughout we will focus on lattice simplices.
- **If 0** \in Δ then the **dual** of Δ is given by

$$
\Delta^{^{\vee}}:=\{\textbf{x}\in\mathbb{R}^d\mid \textbf{x}\,\textbf{y}^{\!\mathsf{T}}\leq 1\,\,\text{for all}\,\,\textbf{y}\in\Delta\}\,.
$$

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■ The fundamental parallelepiped of Δ is

$$
\Pi(\Delta) := \left\{ \sum_{i=1}^{d+1} \lambda_i(\mathbf{v}_i, 1) \middle| 0 \leq \lambda_i < 1 \right\} \subseteq \mathbb{R}^{d+1}.
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■ (Batyrev and Hofscheier):

$$
\Lambda(\Delta):=\left\{\lambda=(\lambda_1,\ldots,\lambda_{d+1})\,\,\Bigg|\,\,\sum_{i=1}^{d+1}\lambda_i(\textbf{v}_i,1)\in\Pi(\Delta)\cap\mathbb{Z}^{d+1}\right\}.
$$

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(Ehrhart) Theory of simplices

$\Lambda(\Delta)\leq (\mathbb{Q}/\mathbb{Z})^{d+1}$

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(Ehrhart) Theory of simplices

$\Lambda(\Delta)\leq (\mathbb{Q}/\mathbb{Z})^{d+1}$ with addition

 $(\lambda_1, \ldots, \lambda_{d+1})+(\lambda'_1, \ldots, \lambda'_{d+1}) = (\{\lambda_1+\lambda'_1\}, \ldots, \{\lambda_{d+1}+\lambda'_{d+1}\}),$

where $\{\cdot\}$ denotes the fractional part of a number.

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(Ehrhart) Theory of simplices

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where $\{\cdot\}$ denotes the fractional part of a number. The h^* -vector of Δ is $h^*(\Delta) = (h_0, h_1, \ldots, h_d)$ where

$$
h_i = \#\{p \in \Pi(\Delta) \cap \mathbb{Z}^{d+1} \mid p_{d+1} = i\}
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 $\mathcal{A} \equiv \mathcal{B} \cup \mathcal{A}$

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(Ehrhart) Theory of simplices

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= # $\{ \lambda \in \Lambda(\Delta) \mid \sum_{j=1}^{d+1} \lambda_j = i \}.$

If $h^*(\Delta)$ is symmetric then Δ is called reflexive.

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Laplacian simplices

Let G be a simple connected graph with *n* vertices. Denote L_G its Laplacian matrix and $\tau(G)$ the number of spanning trees.

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Laplacian simplices

- Let G be a simple connected graph with *n* vertices. Denote L_G its Laplacian matrix and $\tau(G)$ the number of spanning trees.
- Denote $L_G(n)$ the matrix obtained from L_G with the n^{th} column removed and $[L_G(n) | 1]$ the matrix $L_G(n)$ with a coulumn of ones appended.

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Laplacian simplices

- Example connected graph with *n* vertices. Denote L_G its Laplacian matrix and $\tau(G)$ the number of spanning trees.
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Definition (Braun/Meyer, 2017)

The convex hull of the rows of $L_G(n)$, denoted Δ_G , is called the Laplacian simplex associated to G.

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■ The convex hulls of the rows of the Laplacian after removing any column are unimodularly equivalent.

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- The convex hulls of the rows of the Laplacian after removing any column are unimodularly equivalent.
- \blacksquare Vol(Δ _G) = $n \cdot \tau$ (G).

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Reflexive Laplacian simplices

Theorem (Meyer/P, 2018)

Let G be a simple connected graph on n vertices such that Δ_G is reflexive. Then

$$
\Lambda(\Delta_G)=\left\{\frac{\mathbf{x}}{n} \, \Big| \, \overline{\mathbf{x}} \in \text{ker}_{\mathbb{Z}_n}[L(n) \mid 1] \right\}.
$$

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Families of reflexive simplices

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Families of reflexive simplices

(Braun/Meyer, 2017) Trees, odd cycles, and complete graphs yield reflexive Laplacian simplices.

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Families of reflexive simplices

(Braun/Meyer, 2017) Trees, odd cycles, and complete graphs yield reflexive Laplacian simplices.

Definition

To whisker a graph G means to attach an edge and vertex to each existing vertex in G.

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Families of reflexive simplices

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Definition

To whisker a graph G means to attach an edge and vertex to each existing vertex in G. To star a graph G means to attach an additional vertex with every vertex in G.

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Theorem (Meyer/P, 2018)

$$
\blacksquare \text{ Note: } V(\mathcal{W}^*(K_n)) = 2n + 1
$$

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Families of reflexive simplices

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Theorem (Meyer/P, 2018)

■ Note:
$$
V(W^*(K_n)) = 2n + 1, \tau(W^*(K_n)) = (2n + 1)^{n-1}
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Families of reflexive simplices

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Theorem (Meyer/P, 2018)

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$$
V(W^*(K_n)) = 2n + 1, \tau(W^*(K_n)) = (2n + 1)^{n-1},
$$

Vol($\Delta_{W^*(K_n)}$) = $(2n + 1)^n$.

Recall the finite abelian group $\Lambda(\Delta_G)$.

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- Recall the finite abelian group $\Lambda(\Delta_G)$.
- Let A be a finite abelian group and let B be a subgroup. Denote $\hat{A} := \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ the Pontryagin dual of A.

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B^{\circ} = \{ \chi \in \widehat{A} \mid \chi(b) = 1 \text{ for all } b \in B \}.
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- **Duality**
	- Recall the finite abelian group $\Lambda(\Delta_G)$.
	- Let A be a finite abelian group and let B be a subgroup. Denote $A := Hom(A, \mathbb{Q}/\mathbb{Z})$ the Pontryagin dual of A. Denote

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Theorem (Meyer/P, 2018)

Let G be a simple connected graph with n vertices such that the associated Δ_G is reflexive. Then

$$
\Lambda((\Delta_G)^{\vee}) \cong \Lambda(\Delta_G)^{\circ}.
$$

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Future research I: Unimodality

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Future research I: Unimodality

Let G be the graph K_3 with a 7-edge path as a tail. Then

$$
h^*(\Delta_G)=(1,3,3,5,3,5,3,3,1).
$$

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Future research I: Unimodality

Let G be the graph K_3 with a 7-edge path as a tail. Then

$$
h^*(\Delta_G)=(1,3,3,5,3,5,3,3,1).
$$

Open Problem

Classify reflexive Laplacian simplices with unimodal h^* -vector.

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Future research II: Duality

Theorem (Meyer/P, 2018)

Let T_n and K_n denote any tree on n vertices and complete graph respectively. Then

 $\Lambda(\Delta_{K_n})^{\circ} \cong \Lambda(\Delta_{\mathcal{T}_n}).$

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Future research II: Duality

Theorem (Meyer/P, 2018)

Let T_n and K_n denote any tree on n vertices and complete graph respectively. Then

 $\Lambda(\Delta_{K_n})^{\circ} \cong \Lambda(\Delta_{\mathcal{T}_n}).$

Open Problem

Find pairs of graphs (on *n* vertices) (G, H) such that $\Lambda(\Delta_G)^\circ \cong \Lambda(\Delta_H)$.

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Future research II: Duality

Theorem (Meyer/P, 2018)

Let T_n and K_n denote any tree on n vertices and complete graph respectively. Then

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Open Problem

Find pairs of graphs (on *n* vertices) (G, H) such that $\Lambda(\Delta_G)^\circ \cong \Lambda(\Delta_H)$.

Note: A necessary condition is $\tau(G) \cdot \tau(H) = n^{n-2}$.

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Future research $III: h^*$ -vector of the dual

Let Δ_G be reflexive. Denote

$$
h^*(\Delta_G) := \sum_{i=0}^n h_i x^{n-i} y^i.
$$

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Future research $III: h^*$ -vector of the dual

Let Δ_G be reflexive. Denote

$$
h^*(\Delta_G) := \sum_{i=0}^n h_i x^{n-i} y^i.
$$

Open Problem

Is there any relation between $h^*(\Delta_G)$ and $h^*(({\Delta_G})^{\!\vee})?$

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Future research $III: h^*$ -vector of the dual

Let Δ_G be reflexive. Denote

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E Motivation: MacWilliams Duality and MacWilliams Identity.

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Thank You!

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