Additive Codes Associated to Laplacian Simplices

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*Joint with Marie Meyer

Outline

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1. (Ehrhart) Theory of simplices
2. Laplacian simplices
3. Codes and duality

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4. Analysis

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- (**Batyrev and Hofscheier**):

$$\Lambda(\Delta) := \left\{ \lambda = (\lambda_1, \ldots, \lambda_{d+1}) \,\middle|\, \sum_{i=1}^{d+1} \lambda_i(\mathbf{v}_i, 1) \in \Pi(\Delta) \cap \mathbb{Z}^{d+1} \right\}$$

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- The h^*-**vector** of Δ is $h^*(\Delta) = (h_0, h_1, \ldots, h_d)$ where

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- If $h^*(\Delta)$ is symmetric then Δ is called **reflexive**.

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Definition (Braun/Meyer, 2017)

The convex hull of the rows of $L_G(n)$, denoted Δ_G, is called the **Laplacian simplex associated to** G.

Laplacian simplices

- Let $A \in \mathbb{Z}^{n\times n}$ be a square matrix. View A as the $\mathbb{Z}$-module homomorphism $A : \mathbb{Z}_m^n \to \mathbb{Z}_m^n$, $x \mapsto xA$.

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Theorem (Braun/Meyer, 2017)

Let G be a simple connected graph on n vertices. Then

$$\Lambda(\Delta_G) = \left\{ \frac{\mathbf{x}}{n\tau(G)} \;\middle|\; \overline{\mathbf{x}} \in \ker_{\mathbb{Z}_{n\tau(G)}}[L(n) \mid 1] \right\}.$$

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Let G be a simple connected graph on n vertices such that Δ_G is reflexive. Then

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Definition

Let G be a simple connected graph on n vertices such that Δ_G is reflexive. Then $\mathcal{C}(\Delta_G) := \ker_{\mathbb{Z}_n}[L(n) \mid 1] \subseteq \mathbb{Z}_n^n$ is called the **additive code associated to the (reflexive) Laplacian simplex** Δ_G.

Reflexive Laplacian simplices, codes, and duality

Theorem (Meyer/P, 2018)

Let G be a simple connected graph with n vertices such that the associated Δ_G is reflexive. Then

$$\Lambda((\Delta_G)^{\vee}) = \left\{ \frac{\mathbf{x}}{n} \;\middle|\; \overline{\mathbf{x}} \in \mathcal{C}(\Delta_G)^{\perp} \right\}.$$

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- If G and G' are isomorphic then $\mathcal{C}(\Delta_G)$ and $\mathcal{C}(\Delta_{G'})$ are permutation equivalent. The converse is not true!

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For any prime p, there exists a graph G such that $\mathcal{C}(\Delta_G) \subseteq \mathbb{Z}_p^p$ is MDS and has rate (arbitrarily close to) $1/2$.

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Future research

Conjecture

Let G be a graph with p vertices such that Δ_G is reflexive. Show that $\mathcal{C}(\Delta_G)$ is MDS.

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Note: One needs a graph with $2n$ vertices and $(2n)^{n-1}$ spanning trees. A graph that satisfies this is $K_{2,2}$. However $\Delta_{K_{2,2}}$ is not reflexive.

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Let G be a graph such that Δ_G is reflexive. Recall the (finite abelian group) $\Lambda(\Delta_G)$. For $\lambda \in \Lambda(\Delta_G)$ define

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Question

Can one use MacWilliams duality to better understand the h^*-vector of the dual of a simplex?

Thank You!