Dyadic Matrices in Quantum Error-Correction

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Talk is primarily based on

1. M. Martinez, **T. Pllaha**, and C. A. Kelley, "Codes based on dyadic matrices and their generalizations," *Advances in Mathematics of Communications*, 2024.
2. E. Camps-Moreno, G. Matthews, and **T. Pllaha**, "Dyadic matrices and associated quantum codes," submitted.

Definitions

- A matrix $D \in \mathbb{F}_2^{N \times N}$, where $N = 2^n$ is called a **dyadic matrix** if its entries satisfy $D_{\mathbf{x},\mathbf{y}} = D_{\mathbf{0},\mathbf{x}+\mathbf{y}}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n$.

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- The **signature** of a dyadic matrix D, denoted $\sigma(D)$, is the first row $D_{\mathbf{0},\bullet}$ of D.
- For $\mathbf{v} \in \mathbb{F}_2^N$, $D_{\mathbf{v}}$ the corresponding dyadic matrix with signature $\mathbf{v}$, that is, $\sigma(D_{\mathbf{v}}) = \mathbf{v}$.
- A **quasi dyadic matrix** is a matrix with dyadic blocks.
- A code is called (quasi) dyadic if it is the row/null space of a (quasi) dyadic matrix.

For $N = 2^n$, the following hold.

- The set of $N \times N$ dyadic matrices is

$$\mathcal{D}_n = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} \middle| A, B \in \mathcal{D}_{n-1} \right\}$$

  and $\mid \mathcal{D}_n \mid = 2^N$.

- $\mathcal{D}_n \cong \mathbb{F}_2[x_1, \ldots, x_n]/\langle x_1^2 + 1, \ldots, x_n^2 + 1 \rangle$.
- Dyadic matrices are symmetric.
- The zero matrix and the identity matrix in $\mathbb{F}_2^{N \times N}$ are dyadic.
- For any two signatures $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^N$, $D_{\mathbf{u}} + D_{\mathbf{v}} = D_{\mathbf{u}+\mathbf{v}}$.
- Let $D \in \mathcal{D}_n$. Then, either $D^2 = \mathbb{0}_N$ is the zero matrix when $\mathrm{wt}(\sigma(D))$ is even or $D^2 = I_N$ when $\mathrm{wt}(\sigma(D))$ is odd.

Rows and columns will be indexed using elements of $\mathbb{F}_2^n$ via the bijection

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{F}_2^n \leftrightarrow v = 1 + \sum_{i=1}^{n} v_i 2^{i-1} \in [N].$$

Given $M \in \mathbb{F}_2^{N \times N}$, under this bijection, the elements of $\mathbb{F}_2^{n-1} \times \{0\}$ index the "left" half of columns and "top" half of rows, meaning Columns $[N/2]$ and Rows $[N/2]$ of M, whereas the elements of $\mathbb{F}_2^{n-1} \times \{1\}$ index the "right" half of columns and the "bottom" half of rows, meaning Columns $[N/2 : N]$ and Rows $[N/2 : N]$ of M.

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Additionally, the support $\mathrm{supp}(\sigma(D))$ can be viewed both as a subset of $[N]$ and $\mathbb{F}_2^n$.

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Since $D^2 = 0$, we have that $\mathcal{C} = \ker(D)$ satisfies $\mathcal{C}^\perp \subseteq \mathcal{C}$.

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The dyadic code $\mathcal{C} = \mathrm{rs}(D)$ is self-dual if and only if

$$\sum_{\mathbf{x} \in \mathrm{supp}(\sigma(D))} \mathbf{x} \neq \mathbf{0}.$$

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As a consequence

$$\mathrm{Prob}(D \in \mathcal{D}_n \mid \mathrm{rank}(D) = N/2) = \frac{1}{2}\left(1 - \frac{1}{2^n}\right),$$

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Special cases that satisfy the above condition

- $\mathrm{wt}(\sigma(D)) = 2$.
- $\sigma(D) = \mathbf{v} = (\mathbf{a}, \mathbf{b})$ and both $\mathrm{wt}(\mathbf{a}), \mathrm{wt}(\mathbf{b})$ are odd.

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Special cases that do not satisfy the above condition:

- $\mathrm{supp}(\sigma(D))$ is a coset or subspace of $\mathbb{F}_2^n$ (of dim at least 2).

Theorem

Suppose $\mathrm{supp}(\sigma(D))$ is a subspace of $\mathbb{F}_2^n$ of dimension k. Then, for $\mathcal{C} = \mathrm{rs}(D)$ we have $\dim(\mathcal{C}) = 2^{n-k}$.

Proof:

$$d_{\mathbf{x},\mathbf{y}} = d_{\mathbf{0},\mathbf{x}+\mathbf{y}} \implies \mathrm{supp}(D_{\mathbf{x},\bullet}) = \mathbf{x} + \mathrm{supp}(D_{\mathbf{0},\bullet}) \qquad \square$$

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Two notes:

- If $k = 1$ then $\mathcal{C}$ is self-dual.

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Two notes:

- If $k = 1$ then $\mathcal{C}$ is self-dual.
- What if D is supported on a coset $\mathbf{x} + S$? Consider the **dyadic permutation** $P_{\mathbf{x}}$ where $\sigma(P_{\mathbf{x}}) = \mathbf{e}_{\mathbf{x}}$. Then

$$\text{supp}(\sigma(D \cdot P_{\mathbf{x}})) = S.$$

Theorem

Let $D \in \mathcal{D}_n$ be a dyadic matrix.

- If $\mathrm{wt}(\sigma(D))$ is even, then $\mathcal{C} = \mathrm{rs}(D)$ is self-orthogonal.
- If $\mathrm{supp}(\sigma(D))$ is a subspace or a coset of $\mathbb{F}_2^n$ of size 2^k, then $\mathcal{C} = \mathrm{rs}(D)$ is an $[N, 2^{n-k}, 2^k]$ codes whereas $\mathcal{C}^\perp$ is an $[N, N - 2^{n-k}, 2]$ code.

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Let

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$$D = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \text{ such that } A^2 = \mathbb{0}, B^2 = I.$$

For

$$F_1 = \begin{bmatrix} I & B \\ \mathbb{0} & I \end{bmatrix}, F_2 = \begin{bmatrix} B & \mathbb{0} \\ \mathbb{0} & B \end{bmatrix}, \text{ and } F_3 = \begin{bmatrix} I & BA \\ \mathbb{0} & I \end{bmatrix}$$

we have $D = M^{-1}F_1M$ where $M = \Omega F_1F_2F_3$.

Theorem (Calderbank-Shor-Steane (CSS) Construction)

Two classical codes $\mathcal{C}_2 \subseteq \mathcal{C}_1$ with parameters $[n, k_1, d_1], [n, k_2, d_2]$ define a quantum code $\mathcal{Q}(\mathcal{C}_1, \mathcal{C}_2)$ with parameters $[\![n, k_1 - k_2, \geq \min\{d_1, d_2^{\perp}\}]\!]$.

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Common choices of codes:

- Self-orthogonal codes: $\mathcal{C} \subseteq \mathcal{C}^{\perp}$ yields $\mathcal{Q}(\mathcal{C}^{\perp}, \mathcal{C})$ with parameters $[\![n, n-2k, \geq d(\mathcal{C}^{\perp})]\!]$.
- Dual-containing codes: $\mathcal{C}^{\perp} \subseteq \mathcal{C}$ yields $\mathcal{Q}(\mathcal{C}, \mathcal{C}^{\perp})$ with parameters $[\![n, 2k-n, \geq d(\mathcal{C})]\!]$.

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- Dual-containing codes: $\mathcal{C}^\perp \subseteq \mathcal{C}$ yields $\mathcal{Q}(\mathcal{C}, \mathcal{C}^\perp)$ with parameters $[\![n, 2k - n, \geq d(\mathcal{C})]\!]$.
- $\mathcal{C}_1 = \ker(H_1)$ and $\mathcal{C}_2 = \ker(H_2)$, such that $H_1 H_2^\top = \mathbb{0}$ implies $\mathcal{C}_2^\perp \subseteq \mathcal{C}_1$, which yields an $\mathcal{Q}(\mathcal{C}_1, \mathcal{C}_2^\perp)$ with parameters $[\![n, k_1 + k_2 - n, \min\{d_1, d_2\}]\!]$.

Some specific constructions:

- The **hypergraph product codes** constructed from two matrices $A \in \mathbb{F}_2^{m_A \times n_A}, B \in \mathbb{F}_2^{m_B \times n_B}$ and

$$H_1 := \begin{bmatrix} A \otimes I_{m_B} & I_{m_A} \otimes B \end{bmatrix},$$
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  In fact, A, B can take values from any commutative ring.
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- The **(generalized) bicycle codes** constructed from

$$H_1 = \begin{bmatrix} A & B \end{bmatrix}, \quad H_2 = \begin{bmatrix} B & A \end{bmatrix}, \text{ with } AB = BA.$$

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Quantum codes from dyadic and quasi-dyadic matrices

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Quantum codes from dyadic and quasi-dyadic matrices

- Good source of self-dual codes: $[\![n, 0, d]\!]$.
- Good source for short block-length regime: $[\![16, 6, 4]\!]$ with highest achievable distance.
- Distance scales like $\Omega(\sqrt{N})$.
- Expected: improved distance with quasi-dyadic matrices.

The **Schur product** defined as $\mathbf{x} \star \mathbf{y} := (x_1 y_1, \ldots, x_n y_n), \mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n$. Then, the Schur product $\mathcal{C}_1 \star \mathcal{C}_2$ of two codes $\mathcal{C}_1$ and $\mathcal{C}_2$ is the code spanned by $\{\mathbf{c}_1 \star \mathbf{c}_2 \mid \mathbf{c}_i \in \mathcal{C}_i\}$.

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Definition

A pair of codes $\mathcal{C}_2 \subseteq \mathcal{C}_1$ is called **CSS-T pair** if $\mathcal{C}_2 \subseteq \mathcal{C}_1 \cap (\mathcal{C}_1 \star \mathcal{C}_1)^\perp$. A **CSS-T code** is a CSS code $\mathcal{Q}(\mathcal{C}_1, \mathcal{C}_2)$ that corresponds to a CSS-T pair.

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Challenge: $\dim(\mathcal{C}_1 \star \mathcal{C}_1)$ is up to $k_1(k_1 - 1)/2$.
$\mathcal{C}_1 \star \mathcal{C}_1$ blows up fast.
$(\mathcal{C}_1 \star \mathcal{C}_1)^\perp$ is small, and so is $\mathcal{C}_2$.
$\mathcal{C}_2^\perp$ is big and $d_2^\perp$ is small.

Theorem

Let D be a dyadic matrix such that $\mathrm{supp}(\sigma(D))$ is a subspace (or a coset). Then $\mathcal{C} = \mathrm{rs}(D)$ satisfies $\mathcal{C} \star \mathcal{C} = \mathcal{C}$.

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Example

Consider $D \in \mathcal{D}_3$ with $\mathrm{supp}(\sigma(D)) = \{1, 2, 3, 4\}$.
Then $\mathrm{rank}(D) = 2$, and the support forms a subspace.
We have that $\mathcal{C}_1 = \mathrm{rs}(D)$ satisfies $\mathcal{C}_1 \star \mathcal{C}_1 = \mathcal{C}_1$.
This yields an $[\![8, 4, 2]\!]$ CSS-T code.

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This is an optimal code.

Remark

If $\mathcal{C}$ is a code then let us denote $\widehat{\mathcal{C}} = \{(\mathbf{x}, \mathbf{x}) \mid \mathbf{x} \in \mathcal{C}\}$ the concatenation of $\mathcal{C}$ with itself.

If $(\mathcal{C}_1, \mathcal{C}_2)$ is a CSS pair, then $(\widehat{\mathcal{C}_1}, \widehat{\mathcal{C}_2})$ is a CSS-T pair.

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Shout out

E. Berardini, R. Dastbasteh, J. E. Martinez, S. Jain, and O. S. Larrarte, "Asymptotically good CSS-T codes exist," 2024. [Online]. Available: https://arxiv.org/abs/2412.08586.

Thank You!