Decomposition of Clifford Gates

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Outline

- Brief Introduction and Motivation
- Connection between the Clifford Group and Symplectic Group
  - Clifford transvections and symplectic transvections
- A graphical approach for transvection decomposition
- A fast decomposition algorithm
- An example

- Pauli group, or the Heisenberg-Weyl group, on m qubits

$$\mathcal{HW}_N := \{ i^k \mathbf{D}(\mathbf{a},\mathbf{b}) = i^k \mathbf{X}^{a_1}\mathbf{Z}^{b_1} \otimes \cdots \otimes \mathbf{X}^{a_m}\mathbf{Z}^{b_m} \mid (\mathbf{a},\mathbf{b}) \in \mathbb{F}_2^{2m}, k \in \mathbb{Z}_4 \} \subset \mathbb{U}(N)$$

- Hermitian elements in $\mathcal{HW}_N$ are $\mathbf{E} = \mathbf{E}(\mathbf{a},\mathbf{b}) := i^{\mathbf{a}\mathbf{b}^{\mathrm{t}}} \mathbf{D}(\mathbf{a},\mathbf{b})$.
- Every gate $\mathbf{U} \in \mathbb{U}(N)$ can be written as

$$\mathbf{U} = \frac{1}{N} \sum_{\mathbf{v} \in \mathbb{F}_2^{2m}} \mathrm{Tr}\big(\mathbf{E}(\mathbf{v})\mathbf{U}\big)\,\mathbf{E}(\mathbf{v})$$

- The *support*[1] of quantum gates:

$$\mathrm{supp}(\mathbf{U}) := \{ \mathbf{E}(\mathbf{v}) \in \mathcal{HW}_N \mid \mathrm{Tr}\big(\mathbf{E}(\mathbf{v})\mathbf{U}\big) \neq 0 \}.$$

[1] Tefjol Pllaha et al. "Un-Weyl-ing the Clifford Hierarchy". In: *Quantum* 4 (Dec. 2020), p. 370.

Introduction and Motivation (II)

- Clifford Group: Gates that fix $\mathcal{HW}_N$ under conjugation.
- Clifford Group is supported in *subgroups* of $\mathcal{HW}_N$
- Support of *standard* Clifford gates, that is, qubit permutations, diagonal gates, and (partial) Hadamard gates, can be computed in closed form[2]

Problem: What about the support of a general Clifford gate? What about gates on higher levels of the Clifford hierarchy?

Why do we care? Leverage the information encoded by the support to create efficient measurements.

[2] Pllaha et al., "Un-Weyl-ing the Clifford Hierarchy".

The symplectic group

Clifford gates $\mathbf{G}$ on m qubits corresponds to $2m \times 2m$ symplectic matrices $\mathbf{F}$

- Recall: $\mathbf{F}$ is symplectic iff $\mathbf{F}\boldsymbol{\Omega}\mathbf{F}^{t} = \boldsymbol{\Omega}$, where $\boldsymbol{\Omega} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$

Symplectic transvections: For $\mathbf{v} \in \mathbb{F}_2^{2m}$, $\mathbf{T}_{\mathbf{v}} := \mathbf{I}_{2m} + \boldsymbol{\Omega}\mathbf{v}^{t}\mathbf{v}$.
Corresponding *Clifford transvection*: $\mathbf{G}_{\mathbf{v}} = \frac{1}{\sqrt{2}}(\mathbf{I}_N \pm i\mathbf{E}(\mathbf{v}))$.

Theorem:[3] Every sympletic matrix is a product of symplectic transvections.

Corollary: Every Clifford gate is a product of Clifford transvections.

Goal: Find these transvections.

This would also give the support.

[3] Onorato T. O'Meara. *Symplectic groups*. Vol. 16. Mathematical Surveys. American Mathematical Society, Providence, R.I., 1978, pp. xi+122.

A graphical approach

Let $\mathbf{F} = \mathbf{T}_{\mathbf{v}_1} \cdots \mathbf{T}_{\mathbf{v}_r}$ be a symplectic matrix.

Stack the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ to form a $r \times 2m$ matrix $\mathbf{V}$.

Put $\mathbf{A}(\mathbf{v}_1, \ldots, \mathbf{v}_r) := \mathbf{V}\boldsymbol{\Omega}\mathbf{V}^{\mathrm{t}} = [\langle \mathbf{v}_i, \mathbf{v}_j \rangle]_{i,j}$, which is symmetric and has all-zero diagonal.

Consider $\mathbf{A}_u := \mathtt{triu}(\mathbf{A})$ and put

$$\mathbf{B}(\mathbf{v}_1, \ldots, \mathbf{v}_r) := \sum_{\ell=0}^{r-1} \mathbf{A}_u^{\ell}.$$

Interpretation: $\mathbf{A}$ is the adjacency matrix of the *anti-commutation Pauli graph*, and $\mathbf{B}$ counts *all the directed paths* in this graph.

Theorem: For $\mathbf{F} = \mathbf{T}_{\mathbf{v}_1} \cdots \mathbf{T}_{\mathbf{v}_r}$ we have $\mathbf{F} = \mathbf{I} + \boldsymbol{\Omega}\mathbf{V}^{\mathrm{t}}\mathbf{B}\mathbf{V}$.

Definition: $\widehat{\mathbf{F}} := \boldsymbol{\Omega}(\mathbf{I} + \mathbf{F}) = \mathbf{V}^{\mathrm{t}}\mathbf{B}\mathbf{V} = \sum_{i,j} b_{i,j} \mathbf{v}_i^{\mathrm{t}} \mathbf{v}_j$ is called the *residue matrix* of $\mathbf{F}$.

Decomposition

Theorem: In most cases, for a symplectic $\mathbf{F}$

1. there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{P}\widehat{\mathbf{F}}\mathbf{P}^{t} = \begin{bmatrix} \mathbf{B}^{-t} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$
2. $\mathbf{F} = \mathbf{T}_{\mathbf{v}_1} \cdots \mathbf{T}_{\mathbf{v}_r}$, where $\mathbf{v}_i$ is row i of $\mathbf{P}\widehat{\mathbf{F}}$.

Example: For the Clifford $CNOT$ we have $\mathbf{F} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $\widehat{\mathbf{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

First: $\mathbf{F} \leftarrow \mathbf{F}\mathbf{T}_{\mathbf{v}_0}$, where $\mathbf{v}_0 = 0010$ (the first non-zero row of $\widehat{\mathbf{F}}$), for which

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } \widehat{\mathbf{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Decomposition (continued)

$\mathbf{P} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ does the job, for which, $\mathbf{P}\widehat{\mathbf{F}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\mathbf{T}_{\mathbf{v}_1} = \mathbf{T}_{0100} \leftrightarrow \frac{1}{\sqrt{2}}(\mathbf{I}_4 + i\,\mathbf{I}_2 \otimes \mathbf{X})$

$\mathbf{T}_{\mathbf{v}_2} = \mathbf{T}_{0110} \leftrightarrow \frac{1}{\sqrt{2}}(\mathbf{I}_4 - i\,\mathbf{Z} \otimes \mathbf{X})$

$\mathbf{T}_{\mathbf{v}_0} = \mathbf{T}_{0010} \leftrightarrow \frac{1}{\sqrt{2}}(\mathbf{I}_4 + i\,\mathbf{Z} \otimes \mathbf{I}_2)$

$$\begin{aligned} CNOT &= \frac{1-i}{\sqrt{2}} \frac{(\mathbf{I} + i\mathbf{I} \otimes \mathbf{X})(\mathbf{I} - i\mathbf{Z} \otimes \mathbf{X})(\mathbf{I} + i\mathbf{Z} \otimes \mathbf{I})}{\sqrt{8}} \\ &= \frac{1}{2}\left(\mathbf{I}_4 + \mathbf{Z} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{X} - \mathbf{Z} \otimes \mathbf{X}\right) \end{aligned}$$

Thank You!