Reconstruction of Multi-user Binary Subspace Chirps

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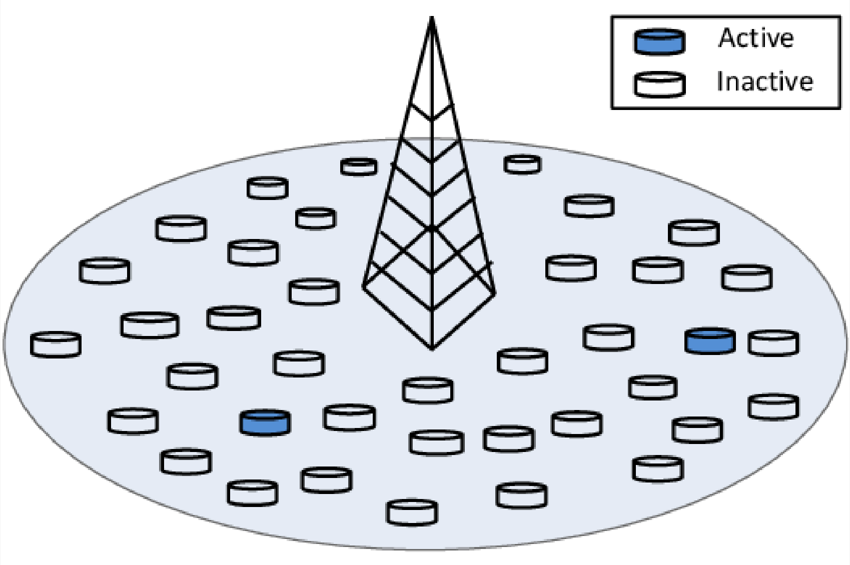
Random Access in mMTC



Framework

- M users, L active users, $L \ll M$.
- Each active user u_ℓ transmits signal $\mathbf{s}_{u_\ell} \in \mathbb{C}^N$.
- Receiver sees

$$\mathbf{s} = \left(\sum_{\ell=1}^{L} \mathbf{c}_\ell \mathbf{s}_{u_\ell} \right) + \mathbf{n}, \quad \mathbf{c}_\ell \in \mathbb{C}, \mathbf{n} \in \mathbb{C}^N.$$

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- **Problem:** Determine $\{u_1, \ldots, u_L\}$ given $\mathbf{s}$.
- **Threshold Decoder:** Declare user u active if $|\mathbf{s}^\dagger \mathbf{s}_u|$ is larger than some threshold.
    - **High complexity:** Requires M measurements!

Binary Chirps

- Fix $m \in \mathbb{N}$ and $\mathbf{S} \in \mathrm{Sym}(m)$ binary symmetric.
- Recall $N = 2^m$. $\mathbb{C}^N$ is indexed with $\mathbb{F}_2^m$ (all vectors, complex or binary, are column vectors).
- Define a unitary matrix $\mathbf{U}_\mathbf{S} \in \mathbb{C}^{N \times N}$ as

$$\mathbf{U}_\mathbf{S}(\mathbf{a}, \mathbf{b}) = \frac{1}{\sqrt{N}} i^{\mathbf{a}^{\mathrm{t}} \mathbf{S} \mathbf{a} + 2\mathbf{b}^{\mathrm{t}} \mathbf{a} \bmod 4}.$$

- A *binary chirp* (BC) is a column $\mathbf{U}_{\mathbf{S},\mathbf{b}}$.
  - BCs can be decoded with $m + 1$ measurements.

Binary Subspace Chirps

Key idea: For $0 \leq r \leq m$, embed all BCs in 2^r dimensions to $N = 2^m$ dimensions and consider all of them jointly.

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- For $\mathbf{P} \in \mathrm{GL}(m)$, $\mathbf{P}^{-\mathrm{t}}$ denotes the inverse transposed.
- Define a unitary matrix $\mathbf{U}_{\mathbf{P},\mathbf{S},r} \in \mathbb{C}^{N \times N}$ as

$$\mathbf{U}_{\mathbf{P},\mathbf{S},r}(\mathbf{a},\mathbf{b}) = \frac{1}{\sqrt{2^r}} i^{(\mathbf{P}^{-1}\mathbf{a})^{\mathrm{t}}\mathbf{S}(\mathbf{P}^{-1}\mathbf{a}) + 2\mathbf{b}^{\mathrm{t}}(\mathbf{P}^{-1}\mathbf{a}) \bmod 4} \cdot \delta_{\mathbf{b},\mathbf{P}^{-1}\mathbf{a},r},$$

  where

$$\delta_{\mathbf{x},\mathbf{y},r} = \begin{cases} 1, & \text{if} \quad (x_{r+1},\ldots,x_m) = (y_{r+1},\ldots,y_m), \\ 0, & \text{else.} \end{cases}$$

- A *binary subspace chirp* (BSSC) is a column $\mathbf{U}_{\mathbf{P},\mathbf{S},\mathbf{b}}$.
- **Note:** Not all choices of $\mathbf{P}, \mathbf{S}$ give different BSSCs.

Parametrization of BSSCs

Theorem

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- Write $H = \mathrm{cs}\,(\mathbf{H}_{\mathcal{I}})$ where $\mathbf{H}_{\mathcal{I}}$ is in CREF and $\mathcal{I}$ is the set of pivots. Then put

$$\mathbf{P} = \mathbf{P}_H = [\mathbf{H}_{\mathcal{I}} \;\; \mathbf{I}_{\widetilde{\mathcal{I}}}].$$

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- $\mathrm{Sym}(r)$ is embedded in $\mathrm{Sym}(m)$ as the upper-left block.
- The total number of BSSCs is

$$2^m \cdot \sum_{r=0}^{m} 2^{r(r+1)/2} \binom{m}{r}_2 = 2^m \cdot \prod_{r=1}^{m} (2^r + 1).$$

  - $|\mathrm{BSSC}|/|\mathrm{BC}| \to 2.384\ldots$

Algebra and Geometry of BSSCs

- The m-qubit *Heisenberg-Weyl* group is

$$\mathcal{HW}_N = \{i^k \mathbf{D}(\mathbf{x}, \mathbf{y}) \mid k = 0, 1, 2, 3, \mathbf{x}, \mathbf{y} \in \mathbb{F}_2^m\} \subset \mathbb{U}(N)$$

where $\mathbf{D}(\mathbf{x}, \mathbf{y}) : \mathbf{e_v} \longmapsto (-1)^{\mathbf{v}\mathbf{y}^{\mathrm{t}}} \mathbf{e_{v+x}}$.

Theorem

(1) There are $\prod_{r=1}^{m}(2^r + 1)$ maximal abelian subgroups if $\mathcal{HW}_N$.

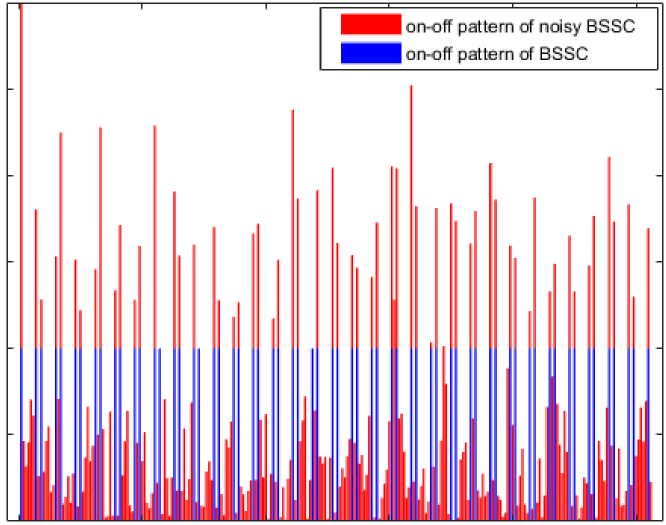
(2) $\mathbf{U}_{\mathbf{P},\mathbf{S},r}$ is the common eigenbase of a unique maximal abelian subgroup of $\mathcal{HW}_N$.

(3) $\mathbf{U}_{\mathbf{P},\mathbf{S},r}$ belongs to the normalizer of $\mathcal{HW}_N$ in $\mathbb{U}(N)$, that is, *Clifford group* Cliff_N.

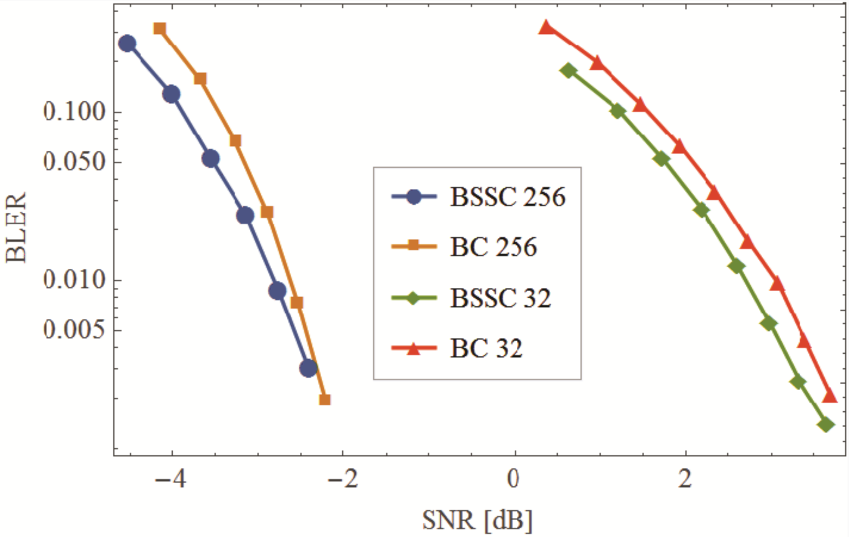
Theorem

Let $\mathbf{w}$ be a rank r BSSCs with on-off pattern determined by $H = \mathrm{cs}(\mathbf{H}_{\mathcal{I}})$. Then $|\mathbf{w}^\dagger \mathbf{D}(\mathbf{0}, \mathbf{y})\mathbf{w}| \neq 0$ iff $\mathbf{y}^{\mathrm{t}} \mathbf{H}_{\mathcal{I}} = \mathbf{0}$.

BSSC vs Noisy BSSC



Error probability of single transmission



Multi BSSCs (no noise)

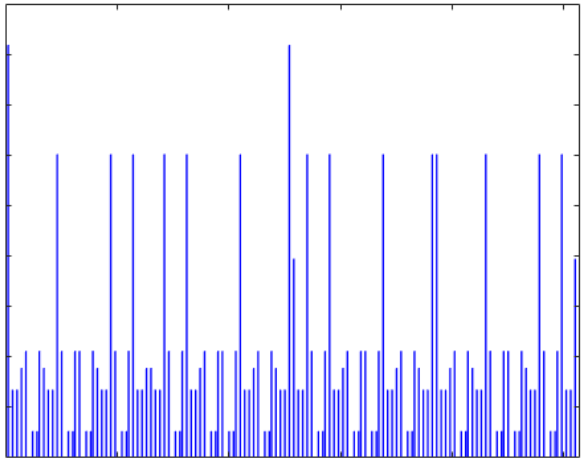
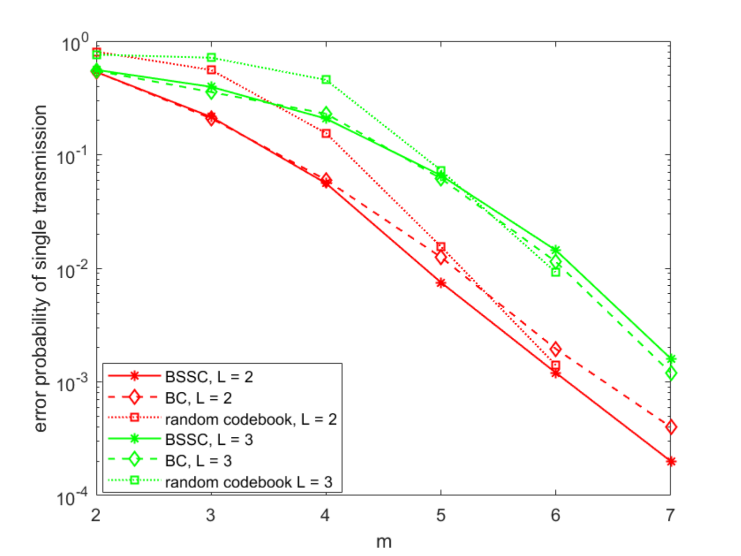


Figure 1: Combination of a rank 2, rank 3, and rank 6 BSSCs in $N = 256$.

Error probability of multiple transmissions



Conclusions

- Codebook of binary chirps expanded to codebook of *binary subspace chirps*.
  - About 2.4 more BSSCs than BCs.
  - Same minimum distance.
- Highly structured codebook.
  - Low complexity algorithm.
  - BSSCs outperform BCs (despite having bigger cardinality and the same minimum distance!).

THANK YOU!