Analysis of syndrome-based iterative decoder failure of QLDPC codes

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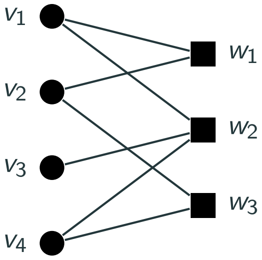
Graph Representation

A linear code $\mathcal{C}$ with parity check matrix H may be represented by a bipartite Tanner graph[1] $G = (V, W; E)$.

- V ("*variable nodes*"), representing codeword coordinates
- W ("*check nodes*"), representing check equations.
- $(v_i, w_j) \in E$ iff $h_{j,i} = 1$.

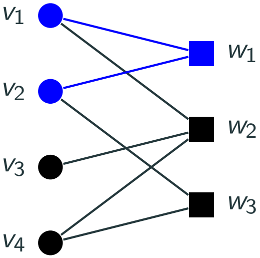
$$H = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$



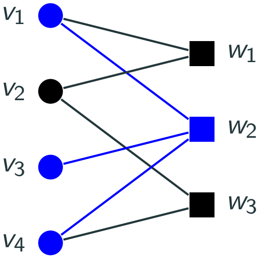
[1] Tanner, "A recursive approach to low complexity codes," 1981

Tanner Graph

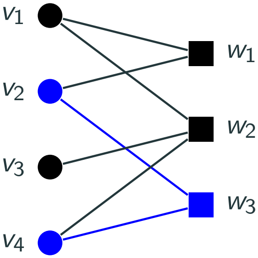
$$\mathbf{x} \in \mathcal{C} \text{ if and only if } H\mathbf{x}^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$$



$w_1 : x_1 + x_2 = 0$



$w_2 : x_1 + x_3 + x_4 = 0$



$w_3 : x_2 + x_4 = 0$

LDPC Codes

A Low Density Parity-Check (LDPC) Code is a linear code with sparse parity-check matrix.

- Low complexity iterative decoding.
- There exists asymptotically good codes.

Stabilizer Codes

$\mathbb{C}^2$	$\leftrightarrow$	$\mathbb{F}_2^2$
$\mathsf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\leftrightarrow$	$(0,0)$
$\mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\leftrightarrow$	$(1,0)$
$\mathsf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\leftrightarrow$	$(0,1)$
$\mathsf{Y} = i\mathsf{XZ}$	$\leftrightarrow$	$(1,0)$
$e = i^c \mathsf{X}^a \mathsf{Z}^b$	$\leftrightarrow$	(a,b)

Stabilizer Codes

$$
\begin{array}{ccc}
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\end{array}
$$

$$
\begin{array}{c|c}
(\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n} & e_1 \otimes \cdots \otimes e_n = i^{c_1}\mathrm{X}^{a_1}\mathrm{Z}^{b_1} \otimes \cdots \otimes i^{c_n}\mathrm{X}^{a_n}\mathrm{Z}^{b_n} \\
\updownarrow & \updownarrow \\
\mathbb{F}_2^{2n} & (e_\mathrm{X}, e_\mathrm{Z}) \equiv (a_1, \ldots, a_n, b_1, \ldots, b_n)
\end{array}
$$

Stabilizer Codes

- Two errors $e \equiv (e_X, e_Z), f \equiv (f_X, f_Z)$ commmute if and only if

$$e \odot f \equiv e_X f_Z^T + e_Z f_X^T = 0$$

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- An $[[n, n-k]]$ stabilizer code is defined by a $k \times 2n$ matrix $H = (H_X \mid H_Z)$ such that

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- Recall:

$$\begin{array}{c|c} (\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n} & e_1 \otimes \cdots \otimes e_n = i^{c_1} X^{a_1} Z^{b_1} \otimes \cdots \otimes i^{c_n} X^{a_n} Z^{b_n} \\ \updownarrow & \updownarrow \\ \mathbb{F}_2^{2n} & (e_X, e_Z) \equiv (a_1, \ldots, a_n, b_1, \ldots, b_n) \end{array}$$

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- The of the codewords of the $[[n, n-k]]$ stabilizer code are eigenvectors of

Calderbank-Shor-Steane (CSS) Codes

A CSS Code is defined by a pair of classical linear codes $C_X, C_Z \subset \mathbb{F}_q^n$ such that $C_X^\perp \subseteq C_Z$.

Two respective parity check matrices H_X and H_Z satisfy $H_Z H_X^T = 0$ and thus the matrix

$$H = \left(\begin{array}{c|c} H_X & 0 \\ 0 & H_Z \end{array}\right)$$

satisfies

$$\begin{pmatrix} H_X \\ 0 \end{pmatrix} \begin{pmatrix} 0 & H_Z^T \end{pmatrix} + \begin{pmatrix} 0 \\ H_Z \end{pmatrix} \begin{pmatrix} H_X^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & H_X H_Z^T \\ H_Z H_X^T & 0 \end{pmatrix} = 0$$

Quantum LDPC Codes: Highlights

- Why quantum LDPC Codes:
  - 2003 (Kitaev): surface code
  - 2013 (Gottesman): quantum LDPC codes achieve fault tolerance with constant overhead
- Good quantum LDPC Codes:
  - 2009 (Tillich & Zémor): $r = c > 0, d \sim \sqrt{n}$
  - 2020-2021: A series of works that broke the $\sqrt{n}$ barrier
  - Nov. 2021: (Panteleev & Kalachev): Asymptotically good codes exist

Syndrome Decoding for CSS Codes

For a CSS code, the syndrome of an error $(e_\mathsf{X}, e_\mathsf{Z})$ is computed as

$$\left(\begin{array}{c|c} H_\mathsf{X} & 0 \\ 0 & H_\mathsf{Z} \end{array}\right) \odot (e_\mathsf{X}, e_\mathsf{Z}) = (H_\mathsf{Z} e_\mathsf{X}^T, H_\mathsf{X} e_\mathsf{Z}^T)$$

$$\equiv (\sigma_\mathsf{X}, \sigma_\mathsf{Z})$$

Thus, C_Z is used to decode X-errors and C_X is used to correct Z errors.

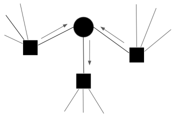
Goal of a syndrome-based decoder: estimate the error pattern $\hat{e}$ whose syndrome $\hat{\sigma}$ matches with the initial input syndrome σ.

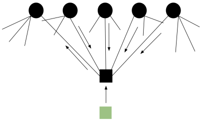
Syndrome-based Iterative Decoder

Input to the decoder: Measured syndrome σ

Process:

- Messages are passed along edges of Tanner graph.
- Nodes wait until they receive messages from all but one neighbor.
- Compute new message to send to remaining neighbor.





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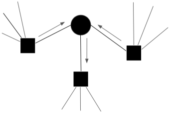
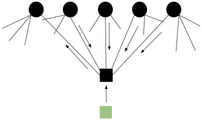


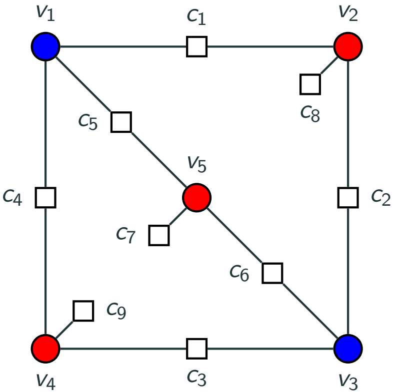


Goal: Estimate the error pattern $\hat{e}$ whose syndrome $\hat{\sigma}$ matches with the initial input syndrome σ.

Gallager-B Syndrome-based Iterative Decoder

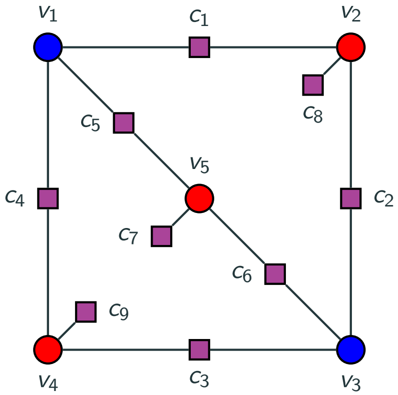
0. Variable nodes are initialized at 0, check nodes are initialized with the input syndrome.
1. The outgoing check node message over an edge is computed as the XOR of extrinsic variable node messages and syndrome input value.
   - Estimated error $\hat{e}$ is determined to be the majority among all incoming check node values at each variable node.
2. The outgoing variable node message is the majority value among incoming extrinsic check node messages.
   - The estimated syndrome is computed as the XOR of all incoming variable node messages.
3. Decoder halts if $\hat{\sigma} = \sigma$ or if ℓ is larger than a threshold.

Example[2]



[2] Raveendran and Vasić, "Trapping Sets of Quantum LDPC Codes," 2021

Example[2]



$$\sigma = (1, 1, 1, 1, 1, 1, 1, 1, 1)$$

[2] Raveendran and Vasić, "Trapping Sets of Quantum LDPC Codes," 2021

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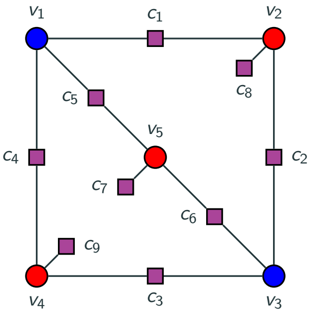

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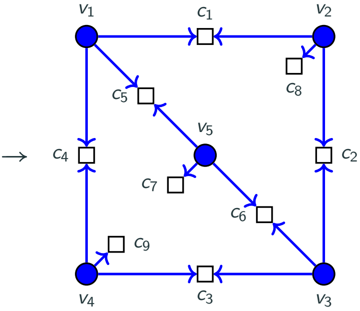

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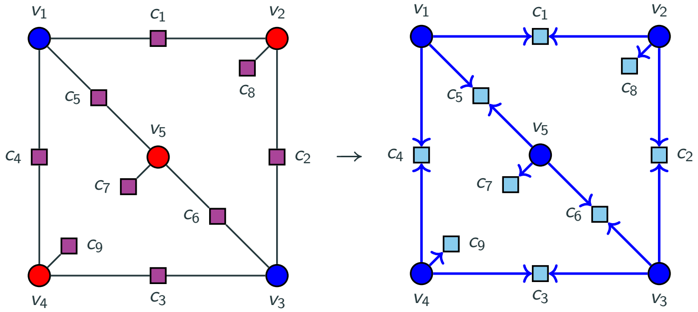

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$\hat{\sigma}^{(1)} = (0, 0, 0, 0, 0, 0, 0, 0, 0)$

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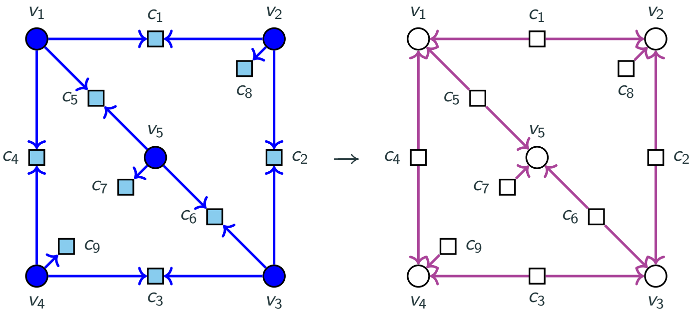

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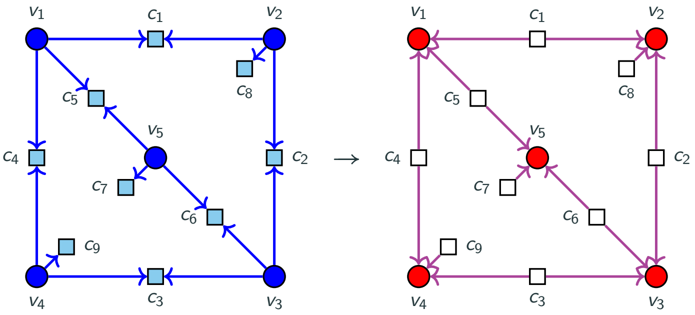

$\hat{\sigma}^{(1)} = (0,0,0,0,0,0,0,0,0)$ $\hat{e}^{(1)} = (1,1,1,1,1)$

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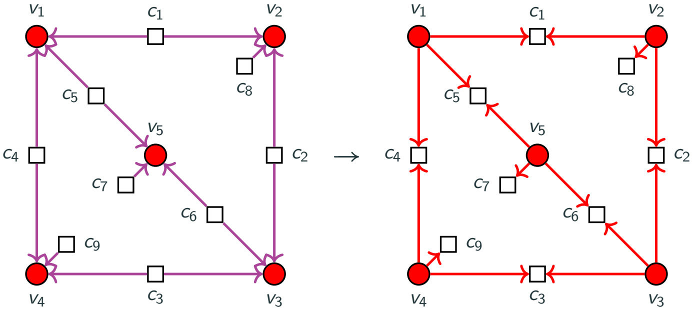

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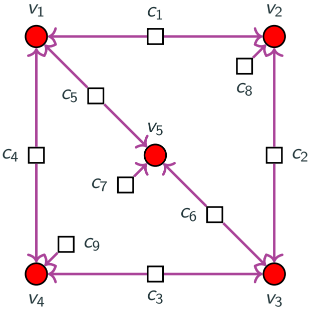
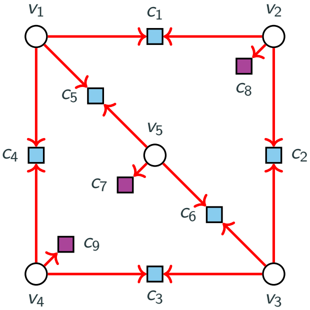


$\hat{e}^{(1)} = (1, 1, 1, 1, 1)$

$\longrightarrow$



$\hat{\sigma}^{(2)} = (0, 0, 0, 0, 0, 0, 1, 1, 1)$

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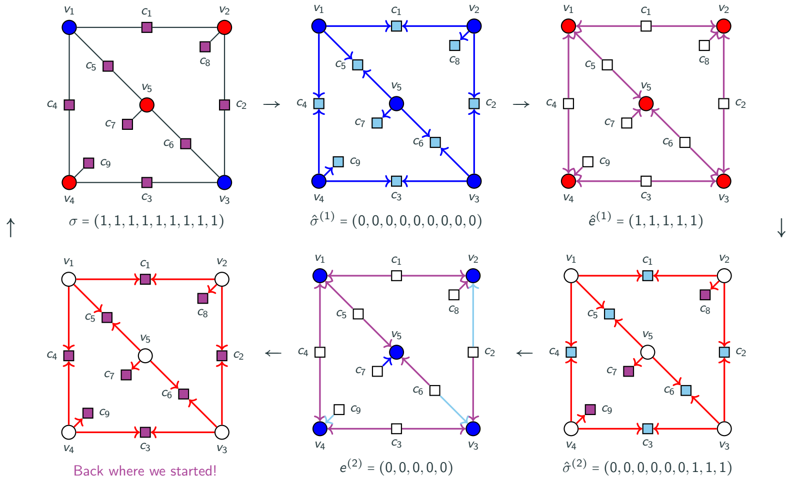
Example[2]



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Trapping Sets

- A check node w_j, for $1 \leq j \leq k$, is eventually correct if there exists $L \in \mathbb{Z}_{\geq 0}$ such that $\hat{\sigma}_j{}^{(\ell)} = \sigma_j$ for all $\ell \geq L$.
- A variable node v_i, for $1 \leq i \leq n$, is said to eventually converge if there exists $L \in \mathbb{Z}_{\geq 0}$ such that $\hat{e}_i^{(\ell)} = \hat{e}_i^{(\ell-1)}$ for all $\ell \geq L$.
- A trapping set for a syndrome-based iterative decoder is a non-empty set of variable nodes $\mathcal{T}$ in a Tanner graph G such that there is a subset of variable nodes $\mathcal{F} \subseteq \mathcal{T}$ that when initially in error result in some subset of check nodes of $\mathcal{N}(\mathcal{T})$ not eventually correct or some variable nodes of $\mathcal{T}$ not eventually converging.
- The graph $\mathcal{T} \cup \mathcal{N}(\mathcal{T})$ is the trapping set graph with respect to $\mathcal{T}$.
- A subset of variable nodes $\mathcal{F}$ that when initially in error result in a trapping set $\mathcal{T}$ is called a failure-inducing set for $\mathcal{T}$.

Related Structures

Nodes in Error	Input Syndrome	Estimated Syndrome	Estimated Error
$\{v_1, v_2, v_3, v_4\}$	$(0,0,0,0,1,1,0,1,1)$	$(0,0,0,0,0,0,1,0,0)$	$\{v_5\}$
$\{v_1, v_2, v_3, v_5\}$	$(0,0,1,1,0,0,1,1,0)$	$(0,0,0,0,0,0,0,0,1)$	$\{v_4\}$
$\{v_1, v_3, v_4, v_5\}$	$(1,1,0,0,0,0,1,0,1)$	$(0,0,0,0,0,0,0,1,0)$	$\{v_2\}$
$\{v_1, v_2, v_4, v_5\}$	$(0,1,1,0,0,1,1,1,1)$	$(0,0,0,0,0,0,0,0,0)$	$\{v_2, v_3, v_4, v_5\}$
		$(0,0,0,0,0,0,0,0,0)$	$\{\}$
		$(1,1,1,1,1,1,0,0,0)$	$\{v_2, v_3, v_4, v_5\}$
		$(1,1,1,1,1,1,0,0,0)$	$\{v_1, v_3\}$
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		$(1,0,0,1,1,0,1,1,1)$	$\{v_1, v_2, v_4, v_5\}$
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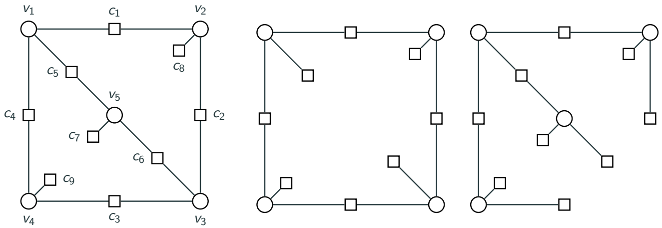
		$(0,0,0,0,0,0,0,0,0)$	$\{v_1, v_2, v_4, v_5\}$



Related Structures: Absorbing Sets

An (a, b) absorbing set $\mathcal{A}$ in a Tanner graph G is a subset of $|\mathcal{A}| = a$ variable nodes such that in the graph $G_{\mathcal{A}}$ induced by $\mathcal{A} \cup \mathcal{N}(\mathcal{A})$ there are b odd degree check nodes and every $v \in \mathcal{A}$ has more even degree than odd degree neighbors in $G_{\mathcal{A}}$.

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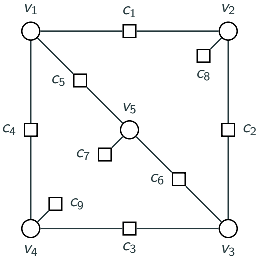
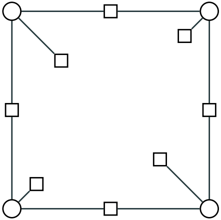
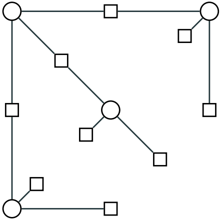


(5,3) absorbing set



(4,4) absorbing set



non absorbing set

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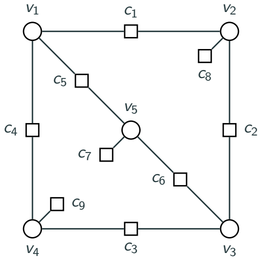

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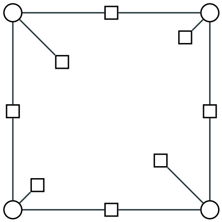

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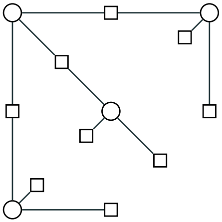

non absorbing set

Question: What relationship is there, if any, between absorbing sets, trapping sets, and failure-inducing sets?

Related Structures: Absorbing Sets

Theorem

Let $\mathcal{A}$ be an (a, b)-absorbing set with $b \geq 1$. Then $\mathcal{A}$ itself is a failure-inducing set and therefore $\mathcal{A}$ is a trapping set.

Proof

- At least one $\sigma_i = 1$.
- The next estimated syndrome $\hat{\sigma} = \overrightarrow{0}$ because
  - Each variable node has strictly more even degree than odd degree check nodes.
  - Even degree nodes send zeros, odd degrees nodes send ones.
  - Majority rules and the estimated error is $\hat{e} = \overrightarrow{0}$.
  - Outgoing variable messages are always 0 because the extrinsic check nodes are at most evenly tied.

Related Structures: Absorbing Sets

Theorem

An (a,0) absorbing set $\mathcal{A}$ a trapping set if and only if $\overrightarrow{1}$ is not a stabilizer.

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- Input syndrome $\sigma = \overrightarrow{0}$ because all CNs have even degree and all VNs are sending 1s.
- syndrome is matched in the first iteration and estimated error is $\hat{e} = \overrightarrow{0}$.
- $e + \hat{e} = \overrightarrow{1}$.

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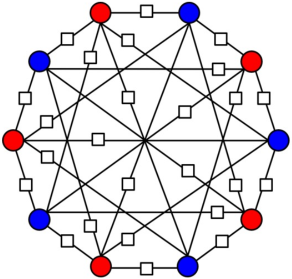
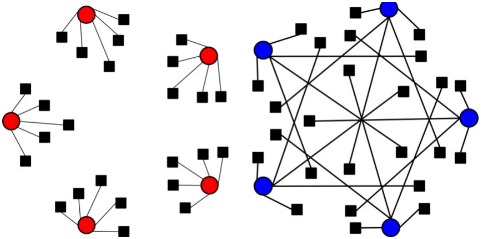
Takeaway

Variable nodes indexed by nonstabilizers with input syndrome $\overrightarrow{0}$ form failure-inducing sets of an $(a, 0)$ absorbing set.

Related $(a, 0)$ absorbing sets: Symmetric Stabilizers[3]



(10,0) absorbing set



Symmetric Split

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Wish list

- Classify all failure-inducing sets within an absorbing set.

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- How to fix absorbing sets?
- What about beyond absorbing sets?

Thank You!