Binary Subspace Chirps

A fast multi-user decoding algorithm

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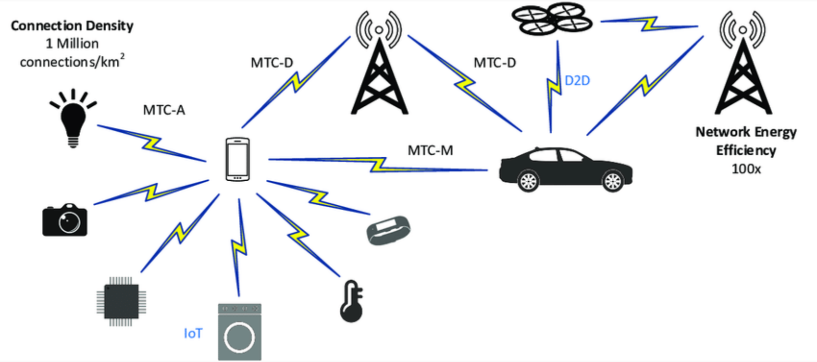
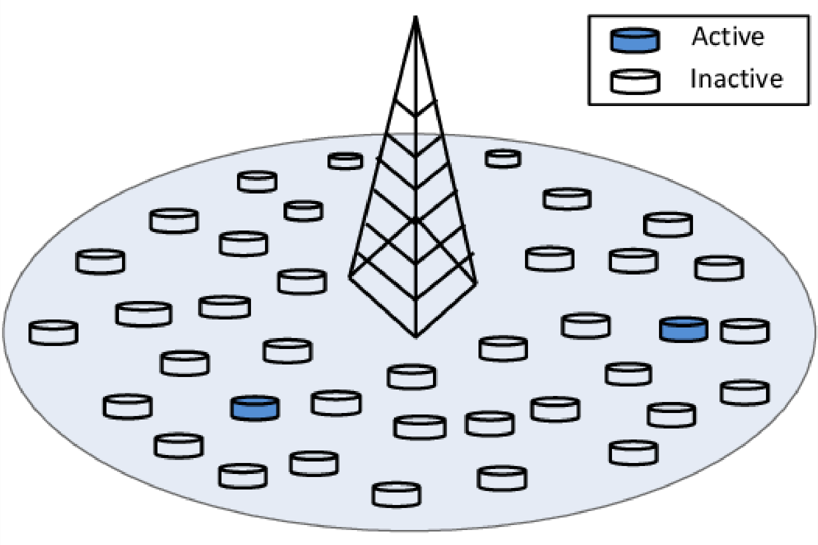
Massive Machine Type Communications (mMTC)



Random Access in mMTC



Framework

- M users, L active users, $L \ll M$.
- Each active user u_ℓ transmits signal $\mathbf{s}_{u_\ell} \in \mathbb{C}^N$.
- Receiver sees

$$\mathbf{s} = \left(\sum_{\ell=1}^{L} \mathbf{c}_\ell \mathbf{s}_{u_\ell} \right) + \mathbf{n}, \quad \mathbf{c}_\ell \in \mathbb{C}, \mathbf{n} \in \mathbb{C}^N.$$

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  - **Design problem:** Construct Φ so that support recovery is possible.
    - **Solution 1:** Restricted Isometric Property (RIP).
    - **Solution 2:** Deterministic sensing matrices.

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  - **High complexity:** Requires M measurements!
- **Key idea:** *Reed-Muller* sequences in $\mathbb{C}^N$ can be decoded with $\log(N) = m$ measurements.

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    - BCs can be decoded with $m + 1$ measurements.

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where

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- **Note:** Not all choices of $\mathbf{P}, \mathbf{S}$ give different BSSCs.

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  - $|\mathrm{BSSC}|/|\mathrm{BC}| \to 2.384\ldots$

Algebra and Geometry of BSSCs

- The m-qubit *Heisenberg-Weyl* group is

$$\mathcal{HW}_N = \{i^k \mathbf{D}(\mathbf{x}, \mathbf{y}) \mid k = 0, 1, 2, 3, \mathbf{x}, \mathbf{y} \in \mathbb{F}_2^m\} \subset \mathbb{U}(N)$$

where $\mathbf{D}(\mathbf{x}, \mathbf{y}) : \mathbf{e_v} \longmapsto (-1)^{\mathbf{v}\mathbf{y}^{\mathrm{t}}} \mathbf{e}_{\mathbf{v}+\mathbf{x}}$.

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(2) $\mathbf{U}_{\mathbf{P},\mathbf{S},r}$ is the common eigenbase of a unique maximal abelian subgroup of $\mathcal{HW}_N$.

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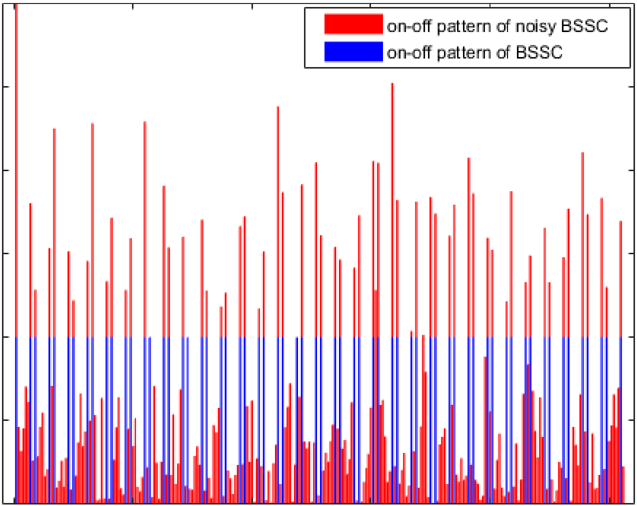
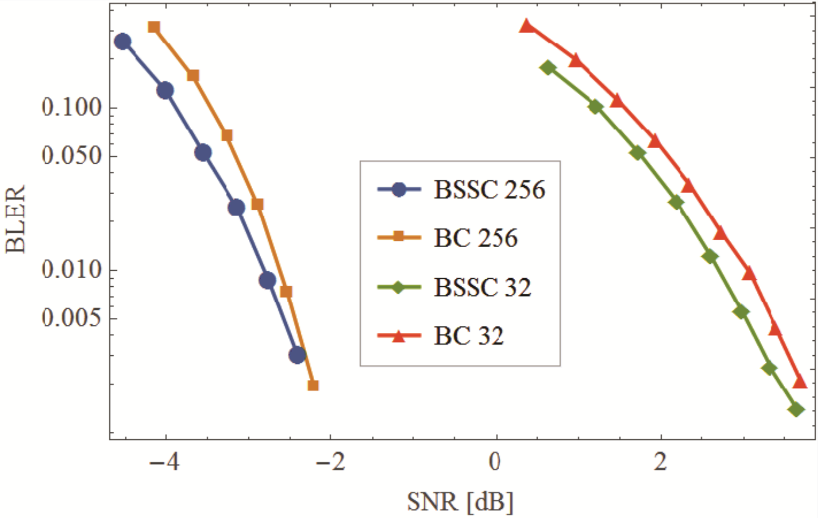
BSSC vs Noisy BSSC



Error probability of single transmission



Multi BSSCs (no noise)

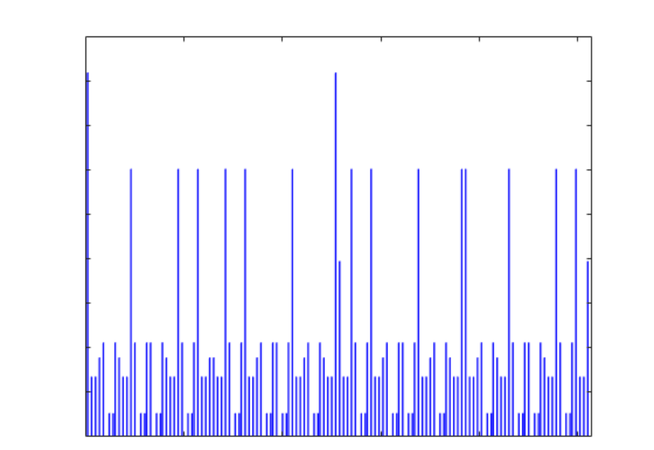
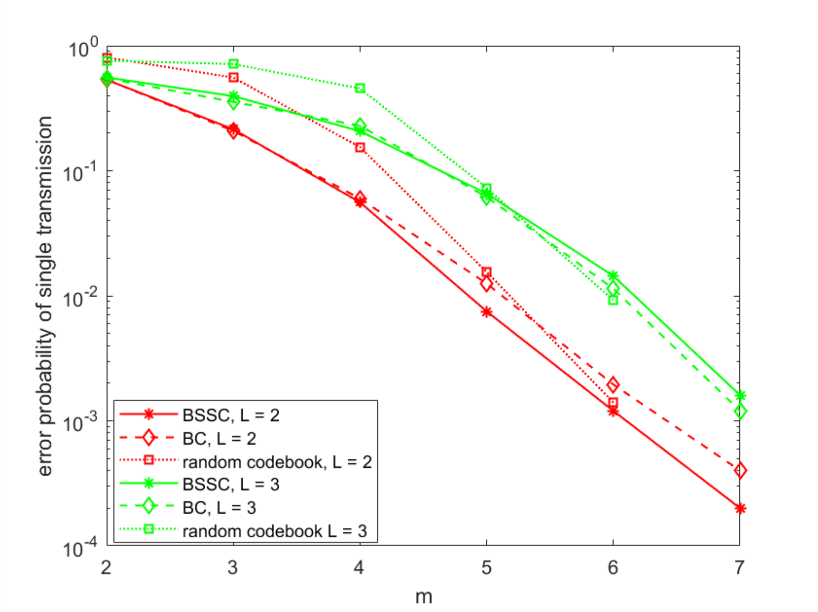
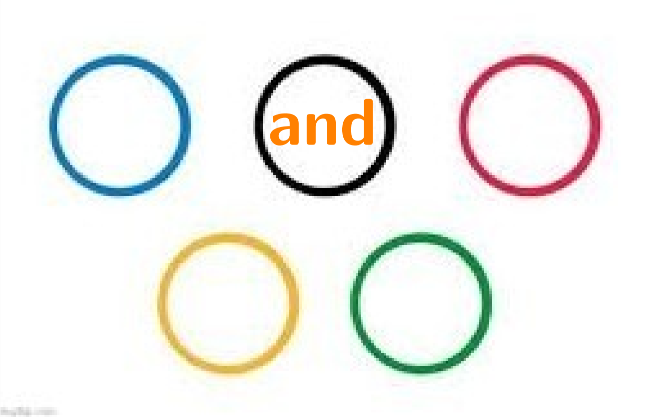


Figure 1: Combination of a rank 2, rank 3, and rank 6 BSSCs in $N = 256$.

Error probability of multiple transmissions



KEEP the DISTANCE



and

WORK from HOME