Un-Weyl-ing the Clifford Hierarchy

Tefjol Pllaha
Joint with **N. Rengaswamy, O. Tirkkonen, and R. Calderbank**

Department of Communications and Networking
Aalto University, Finland

Notation

- Heisenberg-Weyl group $\mathcal{HW}_N$, $N = 2^m$:
  - Pauli matrices: $\mathbf{D}(\mathbf{a}, \mathbf{b})$
  - Hermitian Pauli matrices: $\mathbf{E}(\mathbf{a}, \mathbf{b}) = i^{\mathbf{a}\mathbf{b}^{\mathrm{t}}} \mathbf{D}(\mathbf{a}, \mathbf{b})$
    - $\mathbf{E}^2 = \mathbf{I}_N$ and $\mathbf{E}^\dagger = \mathbf{E}$
  - Projective $\mathcal{HW}_N$: $\mathcal{PHW}_N = \mathcal{HW}_N / \{\pm \mathbf{I}_N, \pm i \mathbf{I}_N\} \cong \mathbb{F}_2^{2m}$.
- Clifford hierarchy $\{\mathcal{C}^{(k)}, k \geq 1\}$:
  - First level $\mathcal{C}^{(1)} = \mathcal{HW}_N$.
  - kth level $\mathcal{C}^{(k)} = \{\mathbf{U} \in \mathbb{U}(N) \mid \mathbf{U} \mathcal{HW}_N \mathbf{U}^\dagger \subset \mathcal{C}^{(k-1)}\}$.
  - Clifford group: $\mathrm{Cliff}_N = \mathcal{C}^{(2)} / \mathbb{U}(1)$.
    - $\mathrm{Cliff}_N / \mathcal{PHW}_N \cong \mathrm{Sp}(2m; 2)$: $\mathbf{G}\mathbf{E}(\mathbf{c})\mathbf{G}^\dagger = \pm \mathbf{E}(\mathbf{c}\mathbf{F})$.

Goals: (1) Better understand the hierarchy.
(2) Characterize $\mathcal{C}^{(3)}$.
(3) (**Motivational question**) What Paulis commute with a given unitary $\mathbf{U}$?

What is known?

U si called **semi-Clifford** if it maps (**under conjugation**) at least one maximal commutative subgroup (MCS) of $\mathcal{HW}_N$ to another MCS.

- $\mathcal{C}^{(k)}$ is made of semi-Cliffords for $\{m = 1, 2, \forall k\}$ and $m = k = 3$
- Zeng et al. (2008) $\mathcal{C}^{(3)}$ is made of semi-Cliffords for all m.
- Gottesman and Mochon (2009) disprove the conjecture.

U si called **generalized** semi-Clifford if it maps the **span** of at least one MCS of $\mathcal{HW}_N$ to the span of another MCS.

- Beigi and Shor (2010) prove that $\mathcal{C}^{(3)}$ is made of generalized semi-Cliffords for all m.

Cui et al., (2017); Rengaswamy et al. (2019) characterize the diagonal hierarchy $\mathcal{C}_d^{(k)}$.

Main tool: the support of a unitary

- $\mathcal{E}_N = \{\mathbf{E}(\mathbf{c}) \mid \mathbf{c} \in \mathbb{F}_2^{2m}\}$ is an orthonormal with respect to

$$\langle \mathbf{M} \mid \mathbf{N} \rangle := \frac{1}{N}\mathrm{Tr}(\mathbf{M}^\dagger \mathbf{N}). \tag{1}$$

- Any unitary $\mathbf{U} \in \mathcal{M}_N(\mathbb{C})$ is a linear combination

$$\mathbf{U} = \sum_{\mathbf{c} \in \mathbb{F}_2^{2m}} \alpha_{\mathbf{c}} \mathbf{E}(\mathbf{c}), \quad \alpha_{\mathbf{c}} = \langle \mathbf{E}(\mathbf{c}) \mid \mathbf{U} \rangle \in \mathbb{C}. \tag{2}$$

Definition

$$\mathrm{supp}(\mathbf{U}) := \{\mathbf{E}(\mathbf{c}) \in \mathcal{HW}_N \mid \alpha_{\mathbf{c}} \neq 0\} \cong \{\mathbf{c} \in \mathbb{F}_2^{2m} \mid \alpha_{\mathbf{c}} \neq 0\}$$

Support of (elementary) Clifford matrices: a detour (I)

Recall: $\mathrm{Cliff}_N/\mathcal{PHW}_N \cong \mathrm{Sp}(2m;2)$: $\mathbf{G}\mathbf{E}(\mathbf{c})\mathbf{G}^\dagger = \pm\mathbf{E}(\mathbf{c}\mathbf{F})$

$$\mathbf{F}_D(\mathbf{P}) = \begin{bmatrix} \mathbf{P} & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{P}^{-\mathrm{t}} \end{bmatrix} \quad \longleftrightarrow \quad \mathbf{G}_D(\mathbf{P}) := |\mathbf{v}\rangle \longmapsto |\mathbf{v}\mathbf{P}\rangle$$

$$\mathbf{F}_U(\mathbf{S}) = \begin{bmatrix} \mathbf{I}_m & \mathbf{S} \\ \mathbf{0}_m & \mathbf{I}_m \end{bmatrix} \quad \longleftrightarrow \quad \mathbf{G}_U(\mathbf{S}) := \mathrm{diag}\left(i^{\mathbf{v}\mathbf{S}\mathbf{v}^{\mathrm{t}} \mod 4}\right)_{\mathbf{v}\in\mathbb{F}_2^m}$$

$$\mathbf{F}_\Omega(r) = \begin{bmatrix} \mathbf{I}_{m|-r} & \mathbf{I}_{m|r} \\ \mathbf{I}_{m|r} & \mathbf{I}_{m|-r} \end{bmatrix} \quad \longleftrightarrow \quad \mathbf{G}_\Omega(r) := (\mathbf{H}_2)^{\otimes r} \otimes \mathbf{I}_{2^{m-r}}$$

Support of (elementary) Clifford matrices: a detour (II)

- For any binary ((typically) invertible) matrix $\mathbf{F}$ put

$$\mathrm{Fix}(\mathbf{F}) := \{\mathbf{v} \in \mathbb{F}_2^{2m} \mid \mathbf{v} = \mathbf{vF}\},$$
$$\mathrm{Res}(\mathbf{F}) := \{\mathbf{v} \oplus \mathbf{vF} \mid \mathbf{v} \in \mathbb{F}_2^{2m}\}.$$

- $\mathbf{F} \in \mathrm{Sp}(2m;2)$ is called a **transvection** if $\dim \mathrm{Res}(\mathbf{F}) = 1$.
- $\mathbf{F}$ is a transvection **iff** there exists (a unique) $\mathbf{v} \in \mathbb{F}_2^{2m}$ such that

$$\mathbf{xF} = \mathbf{x} \oplus \langle \mathbf{v} \,|\, \mathbf{x} \rangle_{\mathrm{s}} \mathbf{v} \quad \text{for all } \mathbf{x},$$

iff $\mathbf{F} = \mathbf{I}_{2m} \oplus \Omega \mathbf{v}^{\mathrm{t}} \mathbf{v} =: \mathbf{T}_{\mathbf{v}}$.

$$\mathbf{T}_{\mathbf{v}} \in \mathrm{Sp}(2m;2) \quad \longleftrightarrow \quad \mathbf{G}_{\mathbf{v}} := \frac{\mathbf{I}_N \pm i\mathbf{E}(\mathbf{v})}{\sqrt{2}} \in \mathrm{Cliff}_N$$

Support of (elementary) Clifford matrices: a detour (III)

Theorem (Callan, 78).

$\mathbf{F} \in \mathrm{Sp}(2m; 2)$ is a product of r or $r+1$ transvections, where $r = \dim \mathrm{Res}(\mathbf{F})$.

Corollary

(1) Any Clifford matrix $\mathbf{G} \in \mathrm{Cliff}_N$ can be written as

$$\mathbf{G} = \mathbf{E}_0 \prod_{n=1}^{k} \frac{\mathbf{I}_N + i\mathbf{E}_n}{\sqrt{2}} = \frac{\mathbf{E}_0}{\sqrt{|S|}} \sum_{\mathbf{E} \in S} \alpha_{\mathbf{E}} \mathbf{E},$$

where $S = \langle \mathbf{E}_1, \ldots, \mathbf{E}_k \rangle$ and $\alpha_{\mathbf{E}} \in \mathbb{C}$.

(2) Any Clifford matrix $\mathbf{G}$ is supported either on a group S or on a coset $\mathbf{E}_0 S$ depending on whether $\mathbf{G}$ has trace or not.

Support of (elementary) Clifford matrices

Proposition

The support of standard Clifford matrices satisfies the following:

(1) $\mathrm{supp}(\mathbf{G}_D(\mathbf{P})) = \mathrm{Res}(\mathbf{P}^{-1}) \times \mathrm{Fix}(\mathbf{P})^{\perp} = \mathrm{Res}(\mathbf{P}^{-1}) \times \mathrm{Res}(\mathbf{P})$.

(2) Let $\mathbf{S} \in \mathrm{Sym}(m)$ and $W = \ker(\mathbf{S}) = \{\mathbf{w} \in \mathbb{F}_2^m \mid \mathbf{wS} = \mathbf{0}\}$. If $\mathrm{Tr}(\mathbf{G}_U(\mathbf{S})) \neq 0$ then $\mathrm{supp}(\mathbf{G}_U(\mathbf{S})) = \{\mathbf{0}\} \times W^{\perp}$. Otherwise $\mathbf{G}_U(\mathbf{S})$ is supported on a coset of $\{\mathbf{0}\} \times W^{\perp}$. As a consequence, the support of diagonal Cliffords is completely characterized by the row/column space of the associated symmetric $\mathbf{S}$.

(3) Let $D_r = \{(\mathbf{x}, \mathbf{0}_{m-r}, \mathbf{x}, \mathbf{0}_{m-r}) \mid \mathbf{x} \in \mathbb{F}_2^r\} \subset \mathbb{F}_2^{2m}$. Then $\mathrm{supp}(\mathbf{G}_\Omega(r)) = (\mathbf{1}_r, \mathbf{0}_{2m-r}) \oplus D_r$, where $\mathbf{1}_r$ denotes the all ones vector of size r. As a consequence, partial Hadamard matrices $\mathbf{G}_\Omega(r)$ are supported on a coset of $\mathrm{Res}(\mathbf{F}_\Omega(r))$.

Example: The CNOT gate

- The CNOT gate is of form $\mathbf{G}_D(\mathbf{P})$, to which corresponds $\mathbf{F}_{\mathrm{CNOT}} = \mathbf{F}_D(\mathbf{P})$, where

$$\mathbf{P} = \mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- $\dim \mathrm{Res}(\mathbf{F}_{\mathrm{CNOT}}) = 2$ and $\mathbf{F}_{\mathrm{CNOT}} = \mathbf{T}_{0010}\mathbf{T}_{0100}\mathbf{T}_{0110}$.
  - Reason for the additional transvection: $\langle \mathbf{v} \,|\, \mathbf{v}\mathbf{F}_{\mathrm{CNOT}} \rangle_{\mathrm{s}} = 0$ for all $\mathbf{v}$, that is, $\mathbf{F}_{\mathrm{CNOT}}$ is **hyperbolic**.
- Put $\mathbf{E}_1 = \mathbf{E}(00, 10), \mathbf{E}_2 = \mathbf{E}(01, 00)$. Then

$$\begin{aligned} \mathrm{CNOT} &= \frac{1-i}{\sqrt{2}} \cdot \frac{(\mathbf{I} + i\mathbf{E}_1)(\mathbf{I} + i\mathbf{E}_2)(\mathbf{I} - i\mathbf{E}_1\mathbf{E}_2)}{\sqrt{8}} \\ &= \frac{1}{2}(\mathbf{I} + \mathbf{E}_1 + \mathbf{E}_2 - \mathbf{E}_1\mathbf{E}_2), \end{aligned}$$

Hermitian Clifford matrices

- Consider the action of $\mathbf{C} \in \mathcal{C}^{(3)}$ on Hermitian Paulis:

$$\varphi_{\mathbf{C}} : \begin{cases} \mathcal{PHW}_N & \xrightarrow{\phi_{\mathbf{C}}} & \mathrm{Cliff}_N & \xrightarrow{\Phi} & \mathrm{Sp}(2m;2) \\ \mathbf{E} & \longmapsto & \mathbf{CEC}^\dagger & \longmapsto & \Phi(\mathbf{CEC}^\dagger) \end{cases}$$

  - $\mathbf{CEC}^\dagger$ is traceless, Hermitian, and involution.

Hermitian Clifford matrices

Theorem

Let $\mathbf{E}_n = \mathbf{E}(\mathbf{c}_n), n = 1, \ldots, k$, be a set of k **independent** Hermitian Pauli matrices. Let also $\mathbf{E}_0 = \mathbf{E}(\mathbf{c}_0)$ be a Hermitian Pauli matrix. Then:

(1) the Clifford matrix

$$\mathbf{G} = \mathbf{E}_0 \prod_{n=1}^{k} \frac{1}{\sqrt{2}}(\mathbf{I} + i\mathbf{E}_n)$$

is Hermitian iff $\mathbf{E}_0$ anticommutes with all $\mathbf{E}_n$ and all $\mathbf{E}_n$ commute with each other.

(2) There exist a quadratic form Q and linear form L such that

$$\mathbf{G} = \frac{1}{\sqrt{2^k}} \sum_{\mathbf{d} \in \mathbb{F}_2^k} i^{Q(\mathbf{d})} \mathbf{E}(L(\mathbf{d})).$$

Semi-Clifford matrices

- A semi-Clifford **C** maps a maximal commutative subgroup (**MSC**) S_1 to some other MSC $S_2 = \mathbf{C} S_1 \mathbf{C}^\dagger$.
  - After a **Clifford correction** we may assume that **C** fixes some MSC.
  - After an **additional** Clifford correction we may assume that **C** fixes **any** MSC.
  - After an **additional** Clifford correction we may assume that **C** fixes **any** MSC **pointwise**.

Theorem (Characterization of semi-Cliffords).

Let **C** be a unitary matrix and S be a MCS. If **C** fixes S pointwise then $\mathrm{supp}(\mathbf{C}) \subset S$. The converse is also true. This property characterizes semi-Clifford matrices up to multiplication by Clifford.

Semi-Clifford matrices

Theorem (Structure of semi-Cliffords).

Let $\mathbf{C} \in \mathcal{C}^{(k)}$ be a unitary matrix that fixes the group of diagonal Paulis $Z_N = \mathbf{E}(\mathbf{0}_m, \mathbf{I}_m)$. Then $\mathbf{C} = \mathbf{D}\mathbf{E}(\mathbf{a}, \mathbf{0})\mathbf{G}_D(\mathbf{P})$, for some diagonal $\mathbf{D} \in \mathcal{C}_d^{(k)}, \mathbf{P} \in \mathrm{GL}(m)$, and $\mathbf{a} \in \mathbb{F}_2^m$.

Proof (Sketch).

- $\mathbf{C} = \mathbf{D}\Pi$, $\mathbf{D} \in \mathcal{C}_d^{(k)}$, Π permutation.
- Diagonals of Z_N are 2nd order Reed-Muller codewords.
- The automorphism group of 2nd order Reed-Muller code is the **general affine** group of maps

$$\mathbf{v} \longmapsto \mathbf{v}\mathbf{P} \oplus \mathbf{a}, \quad \mathbf{P} \in \mathrm{GL}(m), \mathbf{a} \in \mathbb{F}_2^m.$$

The third level $\mathcal{C}^{(3)}$

Lemma

For $\mathbf{C} \in \mathcal{C}^{(3)}$ there exists a Pauli $\widetilde{\mathbf{E}}$ such that $\mathbf{C}\widetilde{\mathbf{E}}\mathbf{C}^\dagger$ is also a Pauli. As a consequence, there exists a Clifford correction $\mathbf{G}$ such that $\mathbf{GC}$ fixes (i.e., commutes with) some Pauli matrix.

Proof (Sketch).

- $$\varphi_{\mathbf{C}} : \begin{cases} \mathcal{PHW}_N & \xrightarrow{\phi_{\mathbf{C}}} & \mathrm{Cliff}_N & \xrightarrow{\Phi} & \mathrm{Sp}(2m;2) \\ \mathbf{E} & \longmapsto & \mathbf{CEC}^\dagger & \longmapsto & \Phi(\mathbf{CEC}^\dagger) \end{cases}$$
- $\ker \varphi_{\mathbf{C}} \subset \mathcal{PHW}_N$ has size 2^k for some $k \geq 0$.
- $G := \mathrm{im}\, \varphi_{\mathbf{C}} \subset \mathrm{Sp}(2m;2)$, of size 2^{2m-k}, acts on $\mathbb{F}_2^{2m} \setminus \{\mathbf{0}\}$.
- There exists an orbit of size 1.
  - **Translation:** There exists $\widetilde{\mathbf{E}}$ that either commutes or anticommutes with all $\mathbf{CEC}^\dagger$.
- Conclude the proof by considering the action of $\widetilde{\mathbf{E}}$ on $\mathbf{C}$.

The third level $\mathcal{C}^{(3)}$

Theorem (Support of third level Cliffords).

Let $\mathbf{C}$ be a unitary matrix from $\mathcal{C}^{(3)}$. Then there exists a Clifford $\mathbf{G}$ such that $\mathbf{GC}$ is supported on a maximal commutative subgroup of $\mathcal{HW}_N$.

Proof (Sketch).

- **Induct on the number of qubits**.
- There exists some Clifford $\mathbf{H}$ such that $\mathbf{HC}$ commutes with some $\mathbf{E} \in \mathcal{HW}_N$.
- Consider $S = \langle \mathbf{E} \rangle$, its normalizer $S^{\perp_s}$, and the resulting $[\![m, m-1]\!]$ stabilizer code.
- Apply induction to the logical (m-1)-qubit operation realized by $\mathbf{HC}$.

The third level $\mathcal{C}^{(3)}$

Corollary (The generalized semi-Clifford Conjecture).

Every $\mathbf{C} \in \mathcal{C}^{(3)}$ is a generalized semi-Clifford matrix.

Future Research (I)

Consider a generic sum of Paulis

$$\mathbf{C} = \sum_{\mathbf{E} \in S} \alpha_{\mathbf{E}} \mathbf{E}.$$

Open Problem

- Characterize $\{\alpha_{\mathbf{E}}\}$ for $\mathbf{C} \in \mathcal{C}^{(k)}$.
- What if, **additionally**, S is MCS?

Conjecture

If S is MCS and $\mathbf{C} \in \mathcal{C}^{(k)}$ then $\{\alpha_{\mathbf{E}}\}$ are **determined** by the kth order Reed-Muller code.

Future Research (II)

- **Recall:** Hermitian Paulis satisfy $\mathbf{E}^2 = \mathbf{I}_N$ and $\mathbf{E}^\dagger = \mathbf{E}$.
- Transvections are **square roots** of Hermitian Paulis, and they **generate** Cliff_N.
- Let $\mathbf{U}$ be a generic unitary such that $\mathbf{U}^2 = \mathbf{I}_N$ and $\mathbf{U}^\dagger = \mathbf{U}$.
  - Its square root $(\mathbf{I}_N + i\mathbf{U})/\sqrt{2}$ is again unitary.

Open Problem

Does there exist a set $\mathcal{U} = \{\mathbf{U} \in \mathbb{U}(N) \mid \mathbf{U}^2 = \mathbf{I}_N \text{ and } \mathbf{U}^\dagger = \mathbf{U}\}$ such that the set of square roots

$$\mathrm{Sqrt}(\mathcal{U}) = \{(\mathbf{I}_N + i\mathbf{U})/\sqrt{2} \mid \mathbf{U} \in \mathcal{U}\}$$

generates $\mathcal{C}^{(3)}$?

Conjecture

Let $\mathcal{G} = \{\mathbf{G} \in \mathrm{Cliff}_N \mid \mathbf{G}^2 \in \mathcal{HW}_N \text{ and } \mathbf{G}^\dagger = \mathbf{G}\}$. Then $\mathrm{Sqrt}(\mathcal{G})$ **generates** $\mathcal{C}^{(3)}$.

THANK YOU!